The time-marching method of fundamental solutions for wave equations

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1. Introduction

The development of highly accurate and efficient wave solvers remains an important and challenging research topic in computational physics, even though there are many numerical methods available for solving hyperbolic-type partial differential equations. The wave equations govern many of physical problems such as the stress wave in an elastic solid, water wave propagation in water bodies, and sound wave propagation in a medium, etc. In this paper, we propose a meshless numerical method based on the method of the fundamental solutions (MFS), the method of particular solutions (MPS) and the Houbolt finite difference (FD) scheme to solve multi-dimensional wave equations with irregular domains.

The so-called meshless numerical schemes can be roughly classified into domain-type and boundary-type methods. The domain-type meshless numerical methods such as the smoothed particle hydrodynamics (SPH) [1] and the multi-quadric collocation method [2,3], etc. are well developed for solving partial differential equations. Boundary-type meshless methods such as the MFS [4] and the hyper-singular meshless method (HMM) [5], etc. have also been developed to obtain solutions of homogeneous partial differential equations. In this study, we propose an extended scheme of the MFS for solving hyperbolic-type partial differential equations such as wave equations.

Kupradze and Aleksidze first proposed the MFS in 1964 [6]. Mathon and Johnston in 1977 solved an elliptic-type boundary value problem by the MFS [7]. Furthermore, Golberg used the two-stage MPS–MFS to solve non-homogeneous elliptic-type equations [8,9]. Previous studies focused on solving elliptic-type partial differential equations by the MFS. For time-dependent problems, the MFS was applied to solve the homogeneous or inhomogeneous diffusion problems either by the time-marching MFS [10,11] or by the so-called unified MFS of diffusion fundamental solution [12,13] or by the eigenfunction expansion MFS [14]. These studies also focused on the solution of parabolic-type partial differential equations.

Although the MFS can successfully deal with elliptic and parabolic problems, it is not so easy to directly handle hyperbolic problems such as wave or advection phenomena. In Ref. [15], the Eulerian–Lagrangian method (ELM) is combined with the MFS to develop a meshless solver for the multi-dimensional advection-diffusion problems, which is called the Eulerian–Lagrangian method of fundamental solutions (ELMFS). The ELMFS has also been applied effectively to solve the nonlinear Burgers’ equations [16], the pure advection equations [17] and even the Navier–Stokes equations [18].

Though the MFS is a powerful and meshfree numerical tool for solving partial differential equations, it is still difficult to solve the wave equation by using the MFS. This is because the fundamental
solution of the wave equation always involves the Dirac delta function (or Heaviside step function). When the fundamental solution of the wave equation is used for the implementation of the MFS, we have to face the difficulty of collocating or differentiating the Dirac delta function (or Heaviside step function) with respect to the time domain for building the linear system. This will induce difficult singularity problems for computer calculation by the MFS. Another well-known process for analyzing the wave equation is to transform the physical variables from the time domain into the frequency domain [19,20]. As a result time-dependent problems become boundary value problems, but sometimes it is more difficult to directly capture the transient phenomena of the wave field via this mode decomposition approach.

The D'Alembert formulation is considered to be very effective for avoiding the problems caused by the presence of the delta function in one-dimensional time-space domains [21]. The D'Alembert formula is combined with the decomposition method to obtain the solution of the wave equation in an infinite domain [22]. The ELMFS is combined with the D'Alembert formula for directly solving the one-dimensional wave equation [23]. In a study by Gu et al. [24], the D'Alembert formula can reduce the one-dimensional wave equation to the advection equation. The ELMFS then approximates the solution of the advection equation. In multi-dimensional wave problems, Hadamard's method of descent is also a well-known mathematical formulation for dealing with the free wave problem [21]. Although the D'Alembert formula can handle time-space Cauchy problems, the reduction also causes problems in treating the boundary conditions. For the same reason, the imposition of boundary conditions becomes complicated when using the method of descent.

The proposed FD–MPS–MFS is a kind of meshless method that combines the advantages of both the domain- and the boundary-type meshless methods. The proposed FD–MPS–MFS model transforms the wave equation into a Poisson-type equation with a time-dependent source term. Thus the hyperbolic problem becomes an elliptic boundary value problem. The dual reciprocity boundary element method (DRBEM), which is similar to the coupled MPS–MFS model, is well developed for solving the wave equation [24,25]. However, the DRBEM requires the time-consuming construction of a surface mesh and numerical quadrature. In [26,27], the coupled MPS–MFS meshless schemes are developed to solve non-homogeneous elliptic partial differential equations, while in [28] they are used to solve the inhomogeneous modified Helmholtz equation and the convection-diffusion equation with variable coefficients. Besides, the coupled MPS–MFS is applied to analyze the conductive problems for functionally graded materials [29] and time-dependent partial differential equations [30].

The time-dependent loading of the system equations is handled by the Houbolt FD method [31]. The Euler scheme is selected to deal with the set-up problem of the proposed FD–MPS–MFS model. Then the physical solution is separated into the particular and the homogeneous solutions at each time step. The particular solution is dependent on radial basis functions while the homogeneous solution is dealt with by the Laplace fundamental solution. In the following sections, we will explain the details of the FD–MPS–MFS wave model and the numerical procedure. Six numerical examples are provided for validating the proposed meshless method. All of the numerical results compare well with the analytical solutions or solutions obtained by the finite element method (FEM). From the numerical tests, it is evident that the proposed FD–MPS–MFS wave model is a promising meshless numerical tool for physical applications.

2. Governing equation

The time-dependent wave equation can be written as

\[
\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi, \quad \vec{x} \in \Omega, \quad t > 0,
\]

where \( \Omega \) is the computational domain with boundary \( \partial \Omega \), \( \phi(\vec{x}, t) \) the physical variable, \( c \) the wave speed, \( t \) denotes time and \( \vec{x} \) the space vector. In Cauchy or initial value problems, the initial conditions are described as follows:

\[
\phi(\vec{x}, t=0) = I(\vec{x}),
\]

Fig. 1. The point distribution (a) two-dimensional problem (Prob. 4.3) (b) three-dimensional problem (Prob. 4.3 and Prob. 4.6).
\[
\frac{\partial \phi(\vec{x}, t)}{\partial t} \bigg|_{t=0} = I_D(x),
\]

where the subscripts \(I\) and \(II\) are used to denote first- and second-kind initial conditions, respectively. The boundary conditions are listed as follows:

\[
\phi(\vec{x}, t)|_{\vec{x} \in \partial \Omega} = g_D(\vec{x}, t),
\]

\[
\frac{\partial \phi(\vec{x}, t)}{\partial \vec{n}} \bigg|_{\vec{x} \in \partial \Omega} = g_N(\vec{x}, t),
\]

where \(\vec{n}\) is the unit outward normal vector to the boundary, \(a\) is a function of space and time. The subscripts \(D\) and \(N\) denote Dirichlet- and Neumann-type boundary conditions, respectively.

3. Numerical methods

The governing equation is a Poisson-type equation with time-dependent loading after the time domain is discretized by the FD scheme. The governing equation and boundary condition can be written as follows:

\[
\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}, \quad \vec{x} \in \Omega,
\]

Fig. 2. The evolution of \(E_{\text{RMS}}\) by the proposed meshless method (Prob. 4.1) (a) 121 points with different time increments (b) 225 points with different time increments (c) 441 points with different time increments and (d) \(\Delta t = 10^{-3}\) with different number of points.
where $B(\phi) = b(\tilde{x}, t), \quad \tilde{x} \in \tilde{\Omega}$.

In order to deal with the transient term of the wave equation, the Houbolt FD method is selected to discretize the time domain. The Houbolt method \cite{31,32} is an implicit and unconditionally stable time-integration scheme that can be obtained by the cubic-Lagrange interpolation of the wave potential $\phi$ from level $(n-2)\Delta t$ through to level $(n+1)\Delta t$. The velocity and acceleration of the wave can be approximated as follows:

$$\frac{\partial \phi^{n+1}}{\partial t} \approx \frac{1}{6\Delta t}(11\phi^{n+1} - 18\phi^n + 9\phi^{n-1} - 2\phi^{n-2}),$$

$$\frac{\partial^2 \phi^{n+1}}{\partial t^2} \approx \frac{1}{\Delta t^2}(2\phi^{n+1} - 5\phi^n + 4\phi^{n-1} - \phi^{n-2}).$$

where $\Delta t$ is the time interval and the superscripts of $t$ represent the time level. After discretizing the time domain of Eq. (6) by introducing the Houbolt method in the time operator, we obtain the following Poisson equation:

$$\nabla^2 \phi^{n+1} = \frac{1}{c^2\Delta t^2}(2\phi^{n+1} - 5\phi^n + 4\phi^{n-1} - \phi^{n-2}).$$

In the MPS–MFS, the solution of the problem can be written as

$$\phi^{n+1} = \phi_p^{n+1} + \phi_h^{n+1},$$

where $\phi_p$ is a particular solution that satisfies the non-homogeneous equation and $\phi_h$ the homogeneous solution that satisfies the Laplace equation.

The particular solution $\phi_p$ can be approximated by radial basis functions (RBFs) as

$$\phi_p^{n+1} = \sum_{j=1}^{M} \beta_j^{n+1} F(r_j),$$

where $F(\cdot)$ is the radial basis function, $\beta_j$ the coefficient of the basis function, $M$ the number of the field points and the subscript $j$ denotes the index of the collocation points. Typical distributions of points are presented in Fig. 1(a) and (b) for two- and three-dimensional simulations, respectively.

The function $F(\cdot)$ can be obtained by analytical integration from the following equation:

$$\nabla^2 F(r) = f(r),$$

where $f(r)$ is the radial basis function. The compactly supported RBF (CSRBF) \cite{33} is selected for describing the particular solution in the multi-dimensional space. The CSRBF $f(r)$ can be written as

$$f(r) = \begin{cases} \left(1 - \frac{r}{\lambda}\right)^2, & r \leq \lambda, \quad x \in \mathbb{R}^2, \\ 0, & r > \lambda, \quad x \in \mathbb{R}^2 \\ \end{cases}$$

where $r$ is the distance between the field points and $\lambda$ the compact radius of the CSRBF. The corresponding radial basis function $F(r)$ for the CSRBF can be obtained as follows:

$$F(r) = \begin{cases} \frac{r^4}{16\lambda^2} - \frac{2r^3}{9\lambda} + \frac{r^2}{3}, & r \leq \lambda, \quad x \in \mathbb{R}^2 \\ \frac{13\lambda^2}{144} + \frac{\lambda^2}{12} \ln \left(\frac{r}{\lambda}\right), & r > \lambda, \quad x \in \mathbb{R}^2 \\ \end{cases}$$

In the MFS, the homogeneous solution can be obtained by a linear combination of fundamental solutions

$$\phi_h^{n+1} = \sum_{j=1}^{N_b} \phi_h^{n+1}(x - x_j),$$

where $N_b$ is the number of the boundary points, the subscript $j$ denotes the index of the source point and $G$ the fundamental solution (or called the free-space Green’s function) which can be written as follows:

$$G(|x - x_j|) = \begin{cases} \frac{-1}{2\pi} \ln(|x - x_j|), & \ x \in \mathbb{R}^2 \\ \frac{1}{4\pi|x - x_j|}, & \ x \in \mathbb{R}^2 \\ \end{cases}$$

where $x$ and $x_j$ are the space-location of the field and source points, respectively. According to the above definitions, we can write

$$\sum_{j=1}^{M} \beta_j^{n+1} f(r_j) = \frac{1}{c^2\Delta t^2}[2(\phi_p^{n+1} + \phi_h^{n+1}) - 5\phi^n + 4\phi^{n-1} - \phi^{n-2}].$$

Fig. 3. The evolution of numerical results and analytical solutions (Prob. 4.1) (a) displacement $u$ at $(\frac{4}{5}, \frac{1}{2})$ and (b) flux $\partial u/\partial x$ at $(1, \frac{1}{2})$. 
The system of equations can be written as
\[
\sum_{j=1}^{M} \beta_j^{n+1} \left( f(r_j) - \frac{2F(r_j)}{c^2\Delta t^2} \right) - \sum_{j=1}^{N_b} \alpha_j^{n+1} \frac{2G(\bar{x} - \bar{z}_j)}{c^2\Delta t^2} = \frac{(-5\phi^n + 4\phi^{n-1} - \phi^{n-2})}{c^2\Delta t^2}.
\]
(19)

While boundary operator, and the boundary condition are given by
\[
\sum_{j=1}^{M} \beta_j^{n+1} B(F(r_j)) + \sum_{j=1}^{N_b} \alpha_j^{n+1} B(G(\bar{x} - \bar{z}_j))) \bigg|_{\bar{x}\in\Omega} = b(\bar{x}, t^{n+1}).
\]
(20)

The linear system can be written in the following form:
\[
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix} \begin{bmatrix}
\beta^{n+1} \\
\alpha^{n+1}
\end{bmatrix} = \begin{bmatrix}
S \\
b
\end{bmatrix},
\]
(21)

where the sub-elements of the linear system are
\[
A_1 = f(r) + CF(r), \quad A_2 = CG(\bar{x} - \bar{z}), \quad A_3 = B(F(r)), \quad A_4 = B(G(\bar{x} - \bar{z})), \quad C = \frac{-2}{(c\Delta t)^2}, \\
S = \frac{1}{c^2\Delta t^2}(-5\phi^n + 4\phi^{n-1} - \phi^{n-2})
\]
(22)

Fig. 4. The evolution of $E_{\text{RMS}}$ by MPS–MFS (Prob. 4.2) (a) 121 points with different time increments (b) 225 points with different time increments (c) 441 points with different time increments and (d) $\Delta t = 1.5 \times 10^{-2}$ with different number of points.
To deal with the set-up problem, the Euler explicit scheme is used for taking the subcomponents \( f_{n}^{1}/C_0 \) and \( f_{n}^{2}/C_0 \) into the vector \( S \) as follows:

\[
\begin{align*}
  f_{n}^{1} &= \frac{1}{\Delta t} \frac{d}{d\tilde{x}} \left( \tilde{x} \cdot \theta(\tilde{x}) \right), \quad n \leq 2 \\
  f_{n}^{2} &= \frac{1}{\Delta t} \frac{d}{d\tilde{x}} \left( 2 \tilde{x} \cdot \theta(\tilde{x}) \right),
\end{align*}
\]

(23)

The vectors \( \beta \) and \( \alpha \) can be obtained by solving linear system (21). After the linear system is solved, the solution in the computational domain can be obtained from

\[
\phi_{n+1} = \sum_{j=1}^{M} \beta_{j}^{n+1} F_{j} + \sum_{j=1}^{N} \alpha_{j}^{n+1} G_{j}(\tilde{x} - \xi_{j}).
\]

(24)

4. Numerical experiments

The proposed FD–MPS–MFS numerical method is tested by considering six numerical experiments. These are analyzed and validated to prove the accuracy of the proposed meshless numerical scheme and also to display the advantages of the proposed wave model. We use the root-mean-square error \( E_{RMS} \) to measure the accuracy, which is defined as

\[
E_{RMS} = \sqrt{\frac{1}{N_{r}} \sum_{i=1}^{N_{r}} (\phi_{i,\text{Analytical}} - \phi_{i,\text{Numerical}})^2},
\]

(25)

where \( N_{r} \) is the number of resolution points.

4.1. The 2D membrane vibration problem

In the first example, the time-dependent 2D membrane vibration problem in a simple geometry having analytic solutions is considered to validate the capability of the proposed meshless method. In this problem, the physical field is governed by the two-dimensional wave equation with unit wave speed in the unit square domain \((0, 1) \times (0, 1)\). The initial displacement and velocity can be written as

\[
\begin{align*}
\phi(x, y, t)|_{t=0} &= xy(1-x)(1-y) \quad \text{and} \quad \frac{\partial \phi(x, y, t)}{\partial t} |_{t=0} = 0,
\end{align*}
\]

(26)

while the boundary condition is given by

\[
\begin{align*}
\phi(x, y, t)|_{x=0} &= 0, \quad \phi(x, y, t)|_{x=1} = 0, \quad \phi(x, y, t)|_{y=0} \\
&= 0, \quad \phi(x, y, t)|_{y=1} = 0.
\end{align*}
\]

(27)

Fig. 5. The evolution of numerical results and analytical solutions (Prob. 4.2) (a) the displacement at \( (\frac{1}{2}, \frac{1}{2}) \) and (b) the displacement at \( (0, \frac{1}{2}) \).

Fig. 6. The evolution of normal flux of numerical results and analytical solutions (Prob. 4.2) (a) normal flux at \( (1, \frac{1}{2}) \) and (b) normal flux at \( (\frac{1}{2}, 1) \).
The analytical solution \( \phi \) for this problem is obtained by the method of separation of variables, which yields

\[
\phi(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\sqrt{m^2 + n^2}\pi ct\right).
\]

(28)

In this case, a uniform point distribution is used in all the tests. Fig. 2(a)–(d) depict the dependence of \( \text{ERMS} \) on different time intervals \( \Delta t \), when 121, 225 and 441 collocation points are used. From Fig. 2(a)–(c), it can be observed that if the time interval decreases \( \text{ERMS} \) will decrease. Fig. 2(d) reveals that as the number of nodes increases, the errors will decrease. In these tests, the solutions show consistent behavior.

When the number of the collocation points is sufficiently large, the \( \text{ERMS} \) values are almost smaller than \( 10^{-3} \). Fig. 3(a) depicts the displacement history at the domain centre \( (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right) \) by the proposed method. Fig. 3(b) depicts the evolution of the normal flux results at the boundary point \( (x, y) = (1, \frac{1}{2}) \) by the proposed method. These results also compare well with the analytical solution. Hence, it is demonstrated that the numerical results obtained by the proposed meshless method are very accurate for the two-dimensional membrane vibration problem even when very few collocation points are used.

4.2. The 2D membrane vibration problem with a second-kind boundary condition

In the second example, the time-dependent 2D membrane vibration problem with a second-kind initial and boundary conditions in a simple geometry is selected to demonstrate the capability of the proposed scheme. The two-dimensional wave equation is considered with a unit wave propagation speed in the square domain \((0, 1) \times (0, 1)\). The initial conditions can be written as

\[
\phi(x, y, t)|_{t=0} = 0 \quad \text{and} \quad \frac{\partial \phi(x, y, t)}{\partial t}|_{t=0} = (1 - x)\sin(\pi y),
\]

(29)

and the boundary condition is

\[
\frac{\partial \phi(x, y, t)}{\partial x}|_{x=0} = 0, \quad \phi(x, y, t)|_{x=1} = 0, \quad \phi(x, y, t)|_{y=0} = 0, \quad \phi(x, y, t)|_{y=1} = 0.
\]

(30)

Fig. 7. The displacement history of the wave propagation problem (Prob. 4.3) (a) at (0, 5) (b) at (0, 0) (c) at (0, 13) and (d) at (-5, -5).
The analytical solution for this problem is obtained by the method of separation of variables, which yields

$$
\phi(x, y, t) = \frac{8\sin(\pi y)}{cn^3} \sum_{n=0}^{\infty} \frac{\sin(\omega_n \pi c t) \cos((2n + 1) \pi x / 2)}{(2n + 1)^2 \omega_n},
$$

$$
\omega_n = \sqrt{1 + \left(\frac{2n + 1}{2}\right)^2}.
$$

(31)

In this case, a uniform distribution of collocation nodes is selected for the tests. Fig. 4(a)–(d) show the relationship between $E_{RMS}$ and different time intervals $\Delta t$, when 121, 225 and 441 collocation points are used for the calculation. From Fig. 4(a)–(c), it is clear that when the time interval decreases, the errors decrease too. Fig. 5(a) and (b) depict the displacement history at the points $(x, y) = (\frac{1}{2}, \frac{1}{2})$ and $(x, y) = (0, \frac{9}{20})$ which compare very well with the analytical solution. Fig. 6(a) and (b) depict the normal flux evolution at the points $(x, y) = (1, \frac{1}{2})$ and $(x, y) = (\frac{1}{2}, 1)$. They also compare well with the analytical solution.

From the numerical results of Figs. 2 and 4, it is interesting to notice that the error analysis from different time intervals, collocation points and RMS errors can serve as the stability criterion for the proposed method. From Figs. 2(d) and 4(d), it can

![Fig. 8. The evolution of wave propagation problem in irregular domain (Prob. 4.3) (a) $t = 0.1$ (b) $t = 1.5$ (c) $t = 5$ (d) $t = 8$ (e) $t = 15$ and (f) $t = 20.$](image)
be observed that when the number of collocation points increases, the errors decrease. In the sensitive case studies, the numerical solutions obtained by the proposed FD–MPS–MFS model are consistent. The numerical experiments also verify the high accuracy of the Houbolt FD scheme and coupled MPS–MFS model even when very few collocation points are used.

4.3. Wave propagation problem in two-dimensional irregular domain

We next consider a more complicated wave propagation problem in an irregular domain with a smooth edge. The initial hump displacement is selected at the center of the domain. Then the smooth wave front propagates from the domain center to the irregular domain edge with a wave speed of \( c = 1.5 \). The computational domain \( \Omega \) and its boundary \( \partial \Omega \) are defined by the following curve:

\[
\partial \Omega = \{(x, y) | x = R_C \cos(\theta), y = R_C \sin(\theta), 0 \leq \theta < 2\pi \},
\]

where \( R_C \) is defined by

\[
R_C = 10 \left[ \cos(4\theta) + \sqrt{\frac{18}{5} - \sin^2(4\theta)} \right]^{1/3}.
\]
The hump shape of the initial displacement and zero initial velocity are selected as follows:

\[
\phi(x, y, t)|_{t=0} = e^{-\frac{4}{25}(x^2+y^2)} \quad \text{and} \quad \frac{\partial \phi(x, y, t)}{\partial t} \bigg|_{t=0} = 0, \tag{34}
\]

and the fixed boundary condition is

\[
\phi(x, y, t)|_{\partial \Omega} = 0. \tag{35}
\]

In this numerical simulation, the setting of the points is similar to the one presented in Fig. 1 (a). In this case, we used 1205
collocation points (with 60 boundary points) with a time interval \( \Delta t = 8 \times 10^{-3} \). In order to verify the correctness of the FD–MPS–MFS numerical results, we also solved the same problem by the FEM (15,981 nodes and 31,448 linear triangular elements) with a time interval \( \Delta t = 5.38 \times 10^{-3} \). Fig. 7(a)–(d) show the evolution history of the displacement at \((x, y) = (0, 5), (0, 0), (10, 5), (5, 0), (0, 13)\) and \((-5, -5)\), respectively. From Fig. 7, we can observe that the FD–MPS–MFS solutions compared rather well with the FEM results. Fig. 8(a)–(f) depict the evolution of the wave phenomena. In this test, the proposed FD–MPS–MFS model uses very few collocation points to deal with this irregular domain problem and still yields as accurate results as the FEM, which uses a very dense mesh and many nodes.

### 4.4. Wave propagation in two-dimensional L-shaped domain

In an attempt to test the algorithm for a more difficult wave propagation problem we consider an L-shaped domain with non-smooth boundary. The Gaussian hump is initially set at the point \((x, y) = (5, 5)\). Then the wave front propagates from the hump center to the domain edge with wave speed \(c = 2\). The L-shaped computational domain \(\Omega\) and initial condition are displayed in Fig. 9. The initial displacement and velocity are selected as

\[
\phi(x, y, t)|_{t=0} = e^{-3/4(x^2+y^2)} \quad \text{and} \quad \frac{\partial \phi(x, y, t)}{\partial t}|_{t=0} = 0, \tag{36}
\]

and the fixed boundary condition is

\[
\phi(x, y, t)|_{\partial \Omega} = 0. \tag{37}
\]

In this case, we used 1976 collocation points (with 200 boundary points) with a time interval \(\Delta t = 5 \times 10^{-3}\). In order to verify the correctness of the FD–MPS–MFS model, we also solved the same problem by using the FEM (19,429 nodes and 38,344 linear triangular elements). Fig. 10(a)–(d) display the evolution history of the displacement at \((x, y) = (5, 5), (10, 5), (15, 5)\) and \((7.5, 7.5)\), respectively. From Fig. 10, we can observe that the FD–MPS–MFS solutions also compare very well with the FEM results. Fig. 11(a)–(f) depict the evolution of the wave propagation phenomena. In comparing with the FEM, the proposed FD–MPS–MFS uses very few points to correctly simulate the wave propagation phenomenon in the L-shaped domain. Even at the test point \((x, y) = (7.5, 7.5)\) which is near the domain edge, the present results are almost identical with the FEM solutions. For problems with deep gaps, the domain decomposition method [34,35] should be used with the proposed method to ease the difficulties of boundary singularity.

### 4.5. Wave transmission in three-dimensional irregular domain

We next consider a wave vibration simulation in a three-dimensional irregular domain with analytic

The initial potential and zero initial velocity are selected as

\[
\phi(x, y, z, t)|_{t=0} = \sin \left( \frac{\pi x}{8} \right) \sin \left( \frac{\pi y}{8} \right) \sin \left( \frac{\pi z}{8} \right) \quad \text{and} \quad \frac{\partial \phi(x, y, t)}{\partial t}|_{t=0} = 0, \tag{40}
\]

and the boundary condition is obtained from the analytical solution which is

\[
\phi(x, y, z, t) = \sin \left( \frac{\pi x}{8} \right) \sin \left( \frac{\pi y}{8} \right) \sin \left( \frac{\pi z}{8} \right) \cos \left( \sqrt{3} \pi ct \right). \tag{41}
\]

In this simulation, the distribution of point locations is depicted in Fig. 1(b). In this case, we used 1389 collocation points (with 361 boundary points), as well as 3233 collocation points (with 648 boundary points) and a time interval \(\Delta t = 2.5 \times 10^{-3}\). In order to demonstrate the accuracy of our numerical results, we also solved the same problem by the FEM with 5261 nodes, 10,786 nodes, 21,783 nodes and 32,526 nodes, respectively. Fig. 12 depicts the evolution of the maximum absolute error obtained by the FD–MPS–MFS and the FEM. It is found that the maximum absolute error curves by the FD–MPS–MFS (1389 points) and the FEM (32,526 nodes) are almost the same. In the \(E_{\text{max}}\) curve obtained with the FD–MPS–MFS (3233 points), the maximum absolute error values always oscillate near \(10^{-3}\) and are smaller than the FEM results with a larger number of nodes. In Fig. 13(a)–(f), we present the solutions at \(x = \pm 3.2, y = \pm 3.2\) and \(z = \pm 3.2\) in the irregular domain. We present a numerical comparison of the FEM and the proposed method in Table 1. This table lists the maximum absolute error, the computational time and memory loading for assessing the efficiency of the two numerical methods. From these comparisons, we conclude that the proposed FD–MPS–MFS model is a highly accurate and efficient numerical tool for solving the multi-dimensional wave problem with irregular domain even when using very few points.

### 4.6. Wave vibration problem in three-dimensional irregular domain

In the final example, we consider a wave vibration simulation in a three-dimensional irregular domain without analytic
solution. In this simulation, the potential field is governed by the wave equation with wave speed \( c = 2 \) in the irregular domain which is the same as in the numerical example (4.5). The initial potential and zero initial velocity are selected as

\[
\phi(\mathbf{x}, t) \big|_{t=0} = 5 \cos \left( \frac{\pi |\mathbf{x}|}{2 R_c} \right) \quad \text{and} \quad \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \big|_{t=0} = 0,
\]

and the boundary condition is

\[
\phi(\mathbf{x}, t) \big|_{\partial\Omega} = 0.
\]

In this case, we used 1389 collocation points (with 361 boundary points) with a time interval \( \Delta t = 2.5 \times 10^{-3} \). To prove the efficiency and accuracy of the FD–MPS–MFS numerical results, we also solved the same problem with the FEM (32,526 nodes and 181,103 linear tetrahedral elements). Fig. 14(a)–(d) depict the evolution of the displacement at the points \((x, y, z) = (0, 0, 0), (6, 0, 0), (\frac{7}{2}, 0, 0)\) and \((0, -4, 0)\), respectively. From Fig. 14(a)–(d), we can see that the FD–MPS–MFS solutions compare quite well with the FEM results at different points of the domain. Fig. 15(a)–(f) present the profiles at \(x = 0, y = 0\) and \(z = 0\) which depict the evolution of wave transmission phenomena. Although the test points are selected near the domain edge \((x, y, z) = (6, 0, 0)\) and

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**Fig. 13.** The evolution of wave vibration problem in irregular domain (Prob. 4.5) (a) \(t = 0\) (b) \(t = 0.5\) (c) \(t = 1\) (d) \(t = 1.5\) (e) \(t = 2\) and (f) \(t = 2.5\).
These results also compare very well with the FEM results. It is worth to point out that to reach the same accuracy; the present FD–MPS–MFS model only used 1389 collocation points while FEM used 32,526 mesh nodes.

5. Conclusion

In this paper, a novel numerical method based on the MPS, the MFS and the Houbolt FD method is developed to approximate the solutions of multi-dimensional wave equations. The Houbolt method is used to avoid the difficulty of constructing the linear algebraic system of the Cauchy conditions. The resulting partial differential equations are handled by the coupled MPS–MFS model. The proposed meshless model is free from numerical quadrature and mesh generation. Six numerical examples in two- and three-dimensional regular and irregular domains are selected to verify the efficacy of the proposed method. From the numerical results, it can be observed that accurate solutions in irregular domains can be obtained easily by the proposed method with very few collocation points. The numerical results demonstrate the
accuracy, consistency and applicability of the proposed meshless numerical model for multi-dimensional wave equations with regular or irregular or non-smooth geometries.

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