1 Comparison of Kansa's method versus Method of Fundamental Solution (MFS) and Dual Reciprocity Method of Fundamental Solution (MFS-DRM)

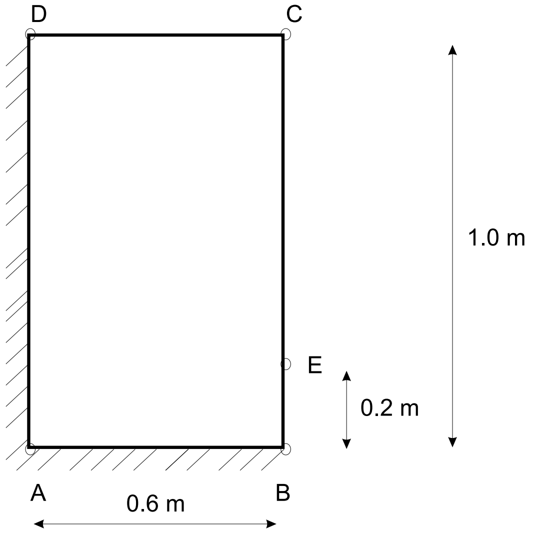
1.1 Introduction

In this work, performances of two most widely used radial basis function based meshless methods (Kansa's and MFS/MFS-DRM) are presented and evaluated. Both were tested on two problems, the first assuming the potential field, and the second with internal source, using three different boundary conditions-Dirichlet, Neumann, and Robin boundary conditions.

1.2 Benchmark test 1

For comparison, NAFEMS benchmark test has been chosen. This test describes 2D thermal problem without internal heat generation and with the following boundary conditions:

- on boundary **AB**: $T = 100^0\mathrm{C}$
- on boundary **DA**: zero flux $\frac{\partial T}{\partial n} = 0$
- on boundaries **BC,CD**: heat transfer with surrounding temperature $0^0\mathrm{C}$



Material properties for this problem are set as follows:

- thermal conductivity $k = 52\,\mathrm{W/mK}$
- heat transfer coefficient $h = 750\,\mathrm{W/mK}$

Combining all listed properties we have:

$$\begin{aligned} -52\,\nabla^2 u &= 0 && \text{in } \Omega && (1)\\ 52\,\frac{\partial u}{\partial n} + 750\,u &= 0 && \text{on } \Gamma_{\mathrm{BC,CD}} \\ u &= 100 && \text{on } \Gamma_{\mathrm{AB}} \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_{\mathrm{AD}} \end{aligned}$$

The results are compared in the reference point at the coordinate (0.6,0.2), where analytical value is calculated to be 18.25375654.

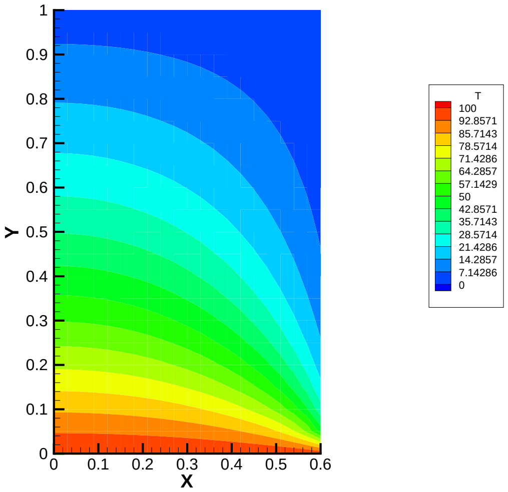


Figure 1: Results of NAFEMS thermal problem

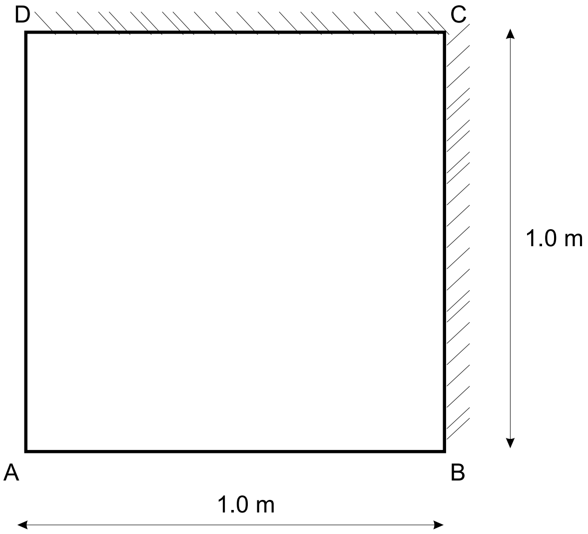
1.3 Benchmark test 2

For this comparison, benchmark test describing 2D thermal problem with internal heat generation and with the following boundary conditions was chosen:

- on boundary **AB, DA** $T = ((1-x)^2 + (1-y)^2)\,^0\mathrm{C}$
- on boundary **BC, CD** zero flux $\frac{\partial T}{\partial n} = 0$

The problem description is

$$\begin{aligned} -\nabla^2 u &= -4 && \text{in } \Omega && (2)\\ u &= (1-x)^2 + (1-y)^2 && \text{on } \Gamma_{\mathrm{AB,DA}} \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_{\mathrm{BC,CD}} \end{aligned}$$



The results are compared across the whole area. The analytical solution for this problem is

$$u = (1 - x)^2 + (1 - y)^2 \tag{3}$$

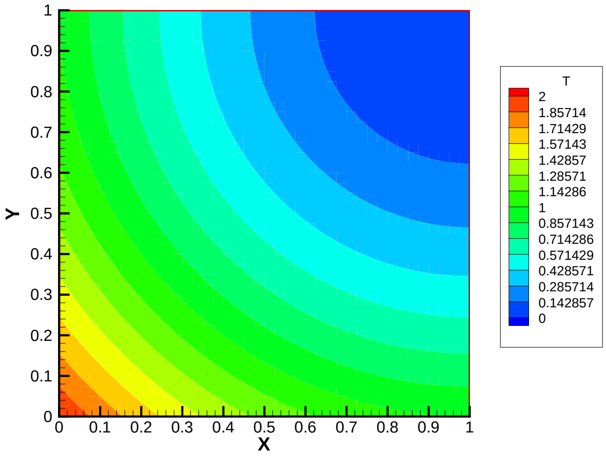


Figure 2: Results of the second NAFEMS thermal problem

1.4 Kansa's method

Kansa's method or Radial basis functions collocation method uses global interpolation direct collocation with radial basis functions (RBS) in order to obtain the solution of partial differential equation. For the problem described above, we have the following formulation: For the elliptic problem, the approximate solution for u is given by

$$\hat{u}(\mathbf{p}) = \sum_{j=1}^{N} \alpha_j \, \varphi_j(\mathbf{p}), \tag{4}$$

where α_j are coefficients to be determined. Similarly we can define the first and the second derivatives of the function $\hat{u}$ in cartesian coordinates

$$\frac{\partial \hat{u}(\mathbf{p})}{\partial \xi} = \sum_{j=1}^{N} \alpha_j \, \frac{\partial \varphi_j(\mathbf{p})}{\partial \xi}, \quad \xi \in x, y \tag{5}$$

and

$$\frac{\partial^2 \hat{u}(\mathbf{p})}{\partial \xi \partial \zeta} = \sum_{j=1}^{N} \alpha_j \, \frac{\partial^2 \varphi_j(\mathbf{p})}{\partial \xi \partial \zeta}, \quad \xi \in x, y, \quad \zeta \in x, y \tag{6}$$

Substituting Eq. (4),(5) and (6) into Eq. (1), we get the formulation for

- Dirichlet boundary condition

$$\sum_{j=1}^{N} \varphi_j \left(\mathbf{p}\right) \alpha_j = T^{\mathcal{D}} \qquad\qquad \mathbf{p} \in \Gamma^{\mathcal{D}}, \tag{7}$$

- Neumann boundary condition

$$\sum_{j=1}^{N} \frac{\partial \varphi_j}{\partial n} \left(\mathbf{p}\right) \alpha_j = T^{\mathcal{N}}, \qquad\qquad \mathbf{p} \in \Gamma^{\mathcal{N}} \tag{8}$$

- Robin boundary condition

$$\sum_{j=1}^{N} \left(k \, \frac{\partial \varphi_j}{\partial n} + h \, \varphi_j \right) \left(\mathbf{p}\right) \alpha_j = h \, T^{\mathcal{R}}_{ref} \qquad\qquad \mathbf{p} \in \Gamma^{\mathcal{R}}, \tag{9}$$

where $\mathbf{p} \in \Gamma$

- Domain interpolation

$$\sum_{j=1}^{N} \left(\frac{\partial^2 \varphi_j}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial y^2} \right) \left(\mathbf{p}\right) \alpha_j = f, \qquad\qquad \mathbf{p} \in \Omega \tag{10}$$

which is a $N \times N$ linear system from where we can calculate the unknowns $\{\alpha_j\}_{j=1}^{N}$. The source f is 0 in the case of Laplace equation.

1.5 MFS

The basic idea of the Method of Fundamental Solution is to solve potential field problem i.e. Laplace equation only by the interpolation over boundary. This is possible by representing solution u as a linear combination of fundamental solutions. Fundamental solution satisfies the solution of Laplace equation

$$\nabla^2 G(r) = 0, \tag{11}$$

from which

$$G(r) = \frac{1}{2\,\pi} \log r, \tag{12}$$

where r is an euclidian distance between two collocation points. Since fundamental solution has singularity when $r = 0$, the approximate solution $\hat{u}$ can be represented by a linear combination of fundamental solutions Eq.(13) of the governing equation with the singularities placed outside the physical domain, where $\hat{\Gamma}$ is the fictitious boundary.

$$\hat{u}(\mathbf{p}) = \sum_{j=1}^{N_\Gamma} \alpha_j \, G(x - x_j, y - y_j), \qquad x_j, y_j \in \hat{\Gamma} \tag{13}$$

Similarly to Eq.(5) and Eq.(6) we can write

$$\frac{\partial \hat{u}(\mathbf{p})}{\partial \xi} = \sum_{j=1}^{N} \alpha_j \, \frac{\partial G_j(\mathbf{p})}{\partial \xi}, \quad \xi \in x, y \tag{14}$$

and

$$\frac{\partial^2 \hat{u}(\mathbf{p})}{\partial \xi \partial \zeta} = \sum_{j=1}^{N} \alpha_j \, \frac{\partial^2 G_j(\mathbf{p})}{\partial \xi \partial \zeta}, \quad \xi \in x, y, \quad \zeta \in x, y \tag{15}$$

1.6 MFS-DRM

In Dual Reciprocity Method of Fundamental Solution the solution $\hat{u}$ is casted into homogenous u_h and particular u_p solution. First, particular solution is solved regardless of boundary conditions using global interpolation of the field. If f is the source, then we can write poisson equation

$$\sum_{j=1}^{N} \alpha_j \, \frac{\partial^2 \varphi_j(\mathbf{p})}{\partial \xi^2} = f, \quad \xi \in x, y \tag{16}$$

where α_j are coefficients to be determined, and f are known values. Once α_j coefficients are known, the particular solution over boundary Γ and domain Ω is constructed as:

$$u_p(\mathbf{p}) = \sum_{j=1}^{N} \alpha_j \, \varphi(\mathbf{p}) \tag{17}$$

When the particular solution u_p is known, the homogenous solution can be calculated as elaborated in section 1.5, only with changed boundary conditions.

- Dirichlet boundary conditions

$$u_h(\mathbf{p}) = T^{\mathcal{D}}(\mathbf{p}) - u_p(\mathbf{p}) \tag{18}$$

- Neumann boundary conditions

$$\frac{\partial u_h}{\partial n}(x,y) = T^{\mathcal{N}}(\mathbf{p}) - \frac{\partial u_p}{\partial n}(\mathbf{p}) \tag{19}$$

- Robin boundary conditions

$$u_h(\mathbf{p}) = h\,T^{\mathcal{R}}(\mathbf{p}) - k\,\frac{\partial u_p}{\partial n}(\mathbf{p}) - h\,u_p(\mathbf{p}) \tag{20}$$

1.7 Comparison

Both methods were tested on the two NAFEMS thermal problems described in Section 1.2 and 1.3. Comparison has been made using several different mesh discretizations, RBSs in Kansa's method and different distances of the fictitious boundary.

1.7.1 Comparison results for the first problem

For Kansa's method the following RBSs has been tested:

- $\varphi = r^3$
- $\varphi = r^5$
- $\varphi = r^7$
- $\varphi = \sqrt{r^2 + c^2}$ with $c = 0.0025$, 0.01, and 0.025

Fictitious boundary distances for MFS are 2, 4, and 6 times of typical mesh distance. Errors for Kansa's method are given in Table 2, errors for MFS are given in Table 1.
In Figure 3 and 4 the influence of boundary points (with constant number of internal points in the case of Kansa's method) on accuracy is shown.

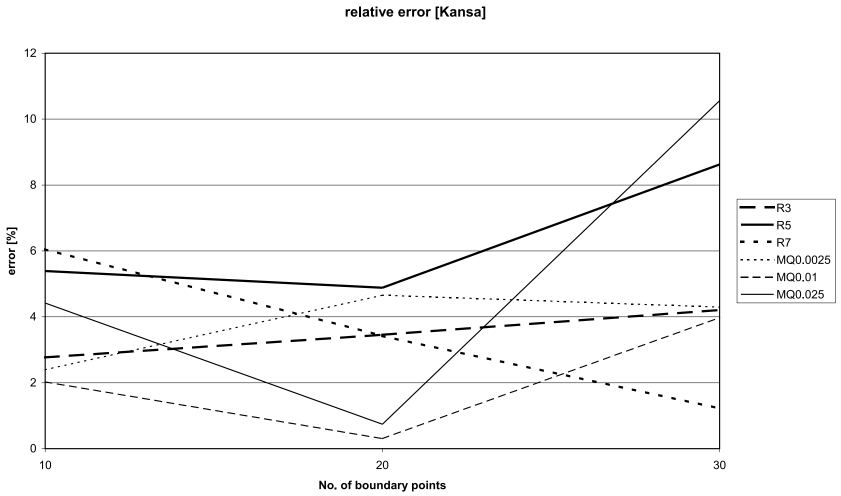


Figure 3: Relative error ($\eta = \frac{\|u_K - u_{ref}\|}{u_{ref}} \cdot 100[\%]$) for different RBSs. (Number marked as No. of boundary points is number of boundary points on each boundary)

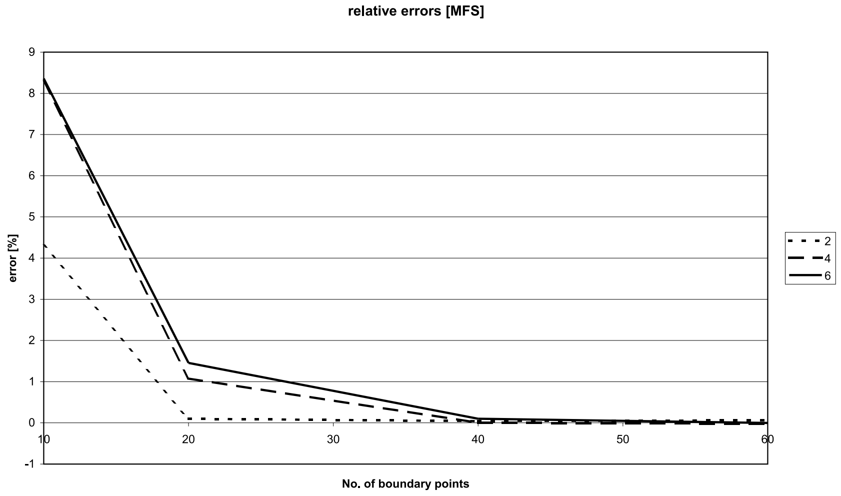


Figure 4: Relative error ($\eta = \frac{\|u_{MFS} - u_{ref}\|}{u_{ref}} \cdot 100[\%]$) for different distances of fictitious boundary from real boundary. (Number marked as No. of boundary points is number of boundary points on each boundary)

$N_{\Gamma,x}$	$N_{\Gamma,y}$	N_{tot}	factor	value	η [%]
10	10	40	2	19.0395	4.3046
20	20	80	2	18.2723	0.1016
40	30	160	2	18.2612	0.0409
60	60	240	2	18.2649	0.0611
10	15	50	2	17.9843	-1.4762
20	30	100	2	18.2421	-0.0639
40	70	220	2	18.2488	-0.2693
60	100	320	2	18.2515	-0.0122
10	10	40	4	19.7658	8.2834
20	20	80	4	18.4503	1.0769
40	30	160	4	18.2543	0.0030
60	60	240	4	18.2505	-0.0181
10	15	50	4	17.9828	-1.4844
20	30	100	4	18.2634	0.0527
40	70	220	4	18.2497	-0.0221
60	100	320	4	18.2518	-0.0106
10	10	40	6	19.7796	8.3592
20	20	80	6	18.5197	1.4568
40	30	160	6	18.2723	0.1018
60	60	240	6	18.2532	-0.0030
10	15	50	6	18.0605	-1.0585
20	30	100	6	18.2584	0.0254
40	70	220	6	18.2501	-0.0201
60	100	320	6	18.2520	-0.0094

Table 1: List of errors for different boundary discretizations. $N_{\Gamma,x}$ is the number of boundary points in x direction, $N_{\Gamma,y}$ is the number of boundary points in y direction,

N_Γ	N_Ω	N_{tot}	RBS	c^2	value	η [%]
10	9	121	R3	-	21.6837	18.7904
20	19	441	R3	-	19.3996	6.2771
8	29	873	R3	-	18.5316	1.5221
10	29	881	R3	-	18.7596	2.7713
20	29	921	R3	-	18.8846	3.4561
30	29	961	R3	-	19.0223	4.2102
10	9	121	R5	-	23.4962	28.7199
20	19	441	R5	-	20.7280	13.5548
8	29	873	R5	-	18.5550	1.6506
10	29	881	R5	-	19.2374	5.3885
20	29	921	R5	-	19.1456	4.8856
30	29	961	R5	-	19.8282	8.6251
10	9	121	R7	-	20.0371	9.7699
20	19	441	R7	-	18.7426	2.6780
8	29	873	R7	-	18.4751	1.2126
10	29	881	R7	-	19.3584	6.0516
20	29	921	R7	-	18.8787	3.4236
30	29	961	R7	-	18.4766	1.2207
20	19	441	MQ	0.0025	16.6076	-9.0182
10	29	881	MQ	0.0025	17.8163	-2.3965
20	29	921	MQ	0.0025	17.4038	-4.6563
30	29	961	MQ	0.0025	17.4700	-4.2936
20	19	441	MQ	0.01	18.3234	0.3815
10	29	881	MQ	0.01	18.6241	2.0289
20	29	921	MQ	0.01	18.3089	0.3020
30	29	961	MQ	0.01	17.5271	-3.9809
20	19	441	MQ	0.025	17.3716	-4.8327
10	29	881	MQ	0.025	19.0602	4.4179
20	29	921	MQ	0.025	18.3891	0.7416
30	29	961	MQ	0.025	16.3260	-10.5609

Table 2: List of errors for different mesh definitions and RBSs.

1.7.2 Comparison results for the second problem

For both methods the following RBSs has been tested:

- $\varphi = r^3$
- $\varphi = r^5$
- $\varphi = r^7$

All distances for fictitious boundary for this problem are set to 3 typical mesh distances in the domain. For the comparison we used different kinds of meshes. Errors for Kansa's method are given in Table 3, errors for MFS are given in Table 4.

N_Γ	N_Ω	N_{tot}	RBS	Abs. error
10	9	121	R3	0.0451
20	19	441	R3	0.0167
8	29	873	R3	0.0164
10	29	881	R3	0.0167
20	29	921	R3	0.0140
30	29	961	R3	0.0095
10	9	121	R5	0.0167
20	19	441	R5	0.0029
8	29	873	R5	0.0014
10	29	881	R5	0.0016
20	29	921	R5	0.0015
30	29	961	R5	0.0010
10	9	121	R7	0.0087
20	19	441	R7	0.0008
8	29	873	R7	0.0014
10	29	881	R7	0.0001
20	29	921	R7	0.0003
30	29	961	R7	0.0002

Table 3: List of errors for Kansa's method for different mesh definitions and RBSs.

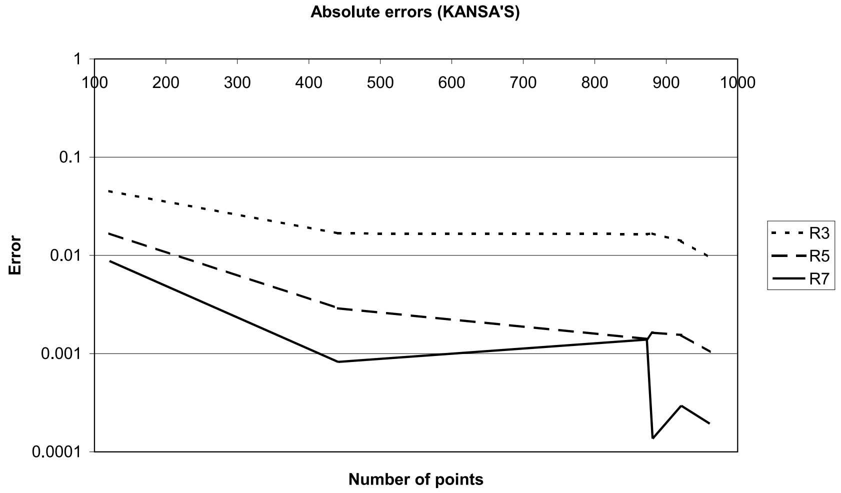


Figure 5: Relative error ($error = \|u_K - u_{ref}\|$) for different RBSs. (Number marked as No. of points is the total number of points)

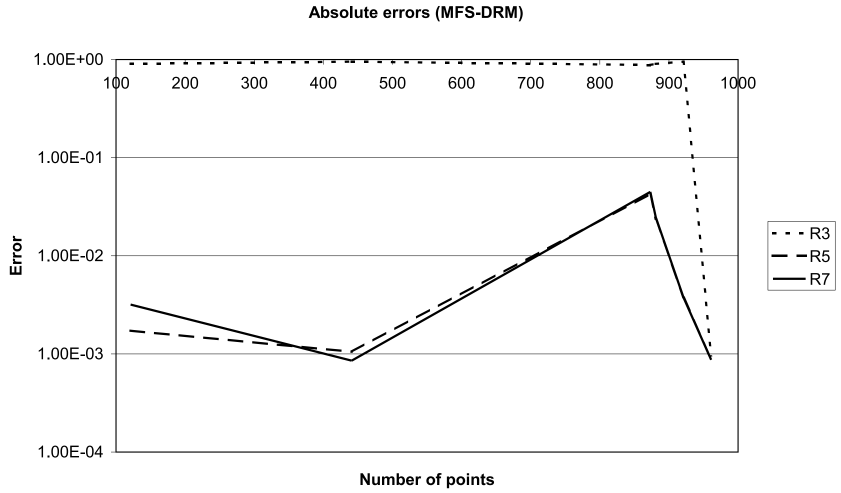


Figure 6: Relative error ($error = \|u_{MFS} - u_{ref}\|$) for different distances of fictitious boundary from real boundary. (Number marked as No. of points is the total number of points)

N_Γ	N_Ω	N_{tot}	RBS	Abs. error
10	9	121	R3	0.9025
20	19	441	R3	0.9506
8	29	873	R3	0.8805
10	29	881	R3	0.9025
20	29	921	R3	0.9506
30	29	961	R3	0.0009
10	9	121	R5	0.0017
20	19	441	R5	0.0011
8	29	873	R5	0.0429
10	29	881	R5	0.0237
20	29	921	R5	0.0038
30	29	961	R5	0.0009
10	9	121	R7	0.0032
20	19	441	R7	0.0008
8	29	873	R7	0.0447
10	29	881	R7	0.0247
20	29	921	R7	0.0037
30	29	961	R7	0.0009

Table 4: List of errors for MFS-DRM for different mesh definitions and RBSs.

1.8 conclusions

The first comparison showed, that MFS is far more precise than Kansa's method. For the case of Laplace equation this is quite understandable, since in MFS interpolation is made using analytical solution of Laplace equation, whereas in Kansa's method RBS functions don't have physical background. Another observation is, that in the case of Kansa's method it is very hard to predict what ratio between boundary and domain points is preferable. The reason for this is, that the definition of the boundary in Kansa's method does not give real information about the problem involved i.e. the definition of the boundary conditions is the same in the case of for example Laplace, Poisson or Navier-Stokes equations. Due to that fact it can happen, that if there is more boundary points than domain points, method does not give us the physically sound solution. In the case of MFS, tests were carried out also for augmented and non augmented formulation. It turns out, that there is difference only on the 6th digit of the calculated value.

Probably more comparable results we get in case of Poisson equation where MFS also needs internal interpolation for particular solution. In the second comparison we showed, that errors are much more comparable. It should be stressed out that in cases with 873,881, and 921 number of points, the typical mesh distance on the boundary is larger than in the domain. That is why the errors are growing in this area. Again we can observe better consistency of the results in the case of MFS-DRM, although the minimum error is lower for some mesh definitions in Kansa's method.

The general conclusion is, that in MFS-DRM the results are far more predictable in terms of grid size.

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