Parametric Curves

There are many useful curves that cannot be described by an equation of the form $y = f(x)$, because $f$ is a function and therefore requires that only one $y$-value be associated with every $x$-value. For example, a complete circle cannot be described by such an equation. In such cases, we can instead describe the curve by parametric equations

$$x = f(t), \quad y = g(t),$$

where the variable $t$ is called a parameter, and the curve defined by these equations is called a parametric curve. For example, a circle of radius $r$ can be defined by the parametric equations

$$x = r \cos t, \quad y = r \sin t.$$

The parameter $t$ is typically restricted to some interval $[a, b]$. The point $x = f(a), y = g(a)$ is then called the initial point of the curve, and the point $x = f(b), y = g(b)$ is called the terminal point of the curve.

Because any functions $f(t)$ and $g(t)$ can be chosen to define the $x$-coordinates and $y$-coordinates, respectively, of points on the curve, there is no requirement that each $x$-value is associated with only $y$-value, as with a curve defined by the equation $y = f(x)$. It follows that any curve in the plane can be defined using parametric equations.

**Example** Construct parametric equations of the form

$$x = f(t), \quad y = g(t)$$

that describe the unit circle.

**Solution** The unit circle is a circle of radius 1 with center at the origin $(0, 0)$. It is described by the equation

$$x^2 + y^2 = 1.$$

Choosing $f(t) = \cos t$ and $g(t) = \sin t$, where $0 \leq t \leq 2\pi$, we find that $x$ and $y$ satisfy this equation and describe the entire circle. If we let $t$ denote time, and let $(x, y) = (f(t), g(t))$ denote the position of a particle at time $t$, then the particle begins at the point $(1, 0)$ (corresponding to $t = 0$) and moves once around the circle in the counterclockwise direction, at constant speed.
An alternative description of this circle is given by the parametric equations

\[ x = \sin e^t, \quad y = \cos e^t, \quad \ln \pi \leq t \leq \ln 3\pi. \]

In this case, a particle whose motion is described by these equations starts at the point \((0, -1)\) and travels once around the circle in the clockwise direction, at steadily increasing speed. □

**Example** Describe the differences between the following sets of parametric equations that represent the curve \(y = x^3\), where \(-\infty < t < \infty\):

1. \(x = t, \quad y = t^3\)
2. \(x = t^2, \quad y = t^6\)
3. \(x = \sin t, \quad y = \sin^3 t\).

**Solution**

1. These equations describe the entire curve \(y = x^3\). A particle whose motion is described by these equations traces the curve from left to right, at constant speed in the \(x\)-direction.
2. These equations describe the portion of the curve in the right-half plane \(x \geq 0\). A particle whose motion is described by these equations traces the curve from right to left as \(t\) increases from \(-\infty\), until \(t = 0\), at which point the particle turns around and retraces the curve from left to right, at constant speed in the \(x\)-direction.
3. These equations describe the portion of the curve for which \(-1 < x < 1\) and \(-1 < y < 1\). A particle whose motion is described by these equations traces the curve from left to right until reaching the point \((1, 1)\), at which point it turns around and retraces the curve from right to left until reaching the point \((-1, -1)\). This process continues forever as \(t\) increases.

□

**Example** Find parametric equations for the astroid \(x^{2/3} + y^{2/3} = 1\).

**Solution** Writing the equation for the astroid as

\[(x^{1/3})^2 + (y^{1/3})^2 = 1,\]

we see that \(x^{1/3}\) and \(y^{1/3}\) can only assume values between \(-1\) and \(1\). Therefore, we can use the identity \(\sin^2 \theta + \cos^2 \theta = 1\) and let \(x^{1/3} = \cos t\) and \(y^{1/3} = \sin t\), which yields the equations

\[x = \cos^3 t, \quad y = \sin^3 t,\]

where \(0 \leq t \leq 2\pi\). □

**Example** Find parametric equations for the ellipse

\[4x^2 + 9y^2 = 36.\]
Solution Rewriting the equation as

\[(2x)^2 + (3y)^2 = 6^2,\]

we see that 2\(x\) and 3\(y\) can only assume values between \(-6\) and 6. Equating \(2x = 6\cos t\) and \(3y = 6\sin t\) yields the equations

\[x = 3\cos t, \quad y = 2\sin t,\]

where \(0 \leq t < 2\pi\). □

Example Sketch the curve described by the parametric equations

\[x = \sin t, \quad y = \sin 2t,\]

where \(0 \leq t \leq 2\pi\).

Solution The curve can be sketched by choosing several values of \(t\) in the interval \([0, 2\pi]\) and computing the corresponding values of \(x\) and \(y\) for each value of \(t\). In Figure 1, the curve is plotted by using Matlab\textsuperscript{TM} to compute \(x\) and \(y\) for \(t = 0, 0.01, 0.02, \ldots\) all the way up to \(2\pi\), plotting the resulting points, and then connecting the points to obtain a smooth curve. □

Example Given a curve defined by the parametric equations

\[x = 3t + 2, \quad y = t - 1,\]

eliminate the parameter \(t\) and obtain a Cartesian equation for the curve.

Solution By a Cartesian equation, we mean an equation of the form \(y = f(x)\) or \(x = f(y)\). In this case, we can obtain either type of equation since both \(x\) and \(y\) are one-to-one functions of \(t\). We choose to obtain an equation of the form \(y = f(x)\). Solving the equation \(x = 3t + 2\) for \(t\), we obtain

\[t = \frac{x - 2}{3}.\]

Substituting this expression for \(t\) into the equation \(y = t - 1\), we obtain the equation

\[y = f(x) = \frac{x - 2}{3} - 1 = \frac{x - 5}{3}.\]

□

Example Given a curve defined by the parametric equations

\[x = \sqrt{t + 1}, \quad y = e^t,\]

where \(t \geq 0\), eliminate the parameter \(t\) and obtain a Cartesian equation for the curve.
Figure 1: Curve defined by the parametric equations $x = \sin t$, $y = \sin 2t$.

**Solution** Since $x$ is a one-to-one function of $t$, we can solve the equation $x = \sqrt{t+1}$ for $t$ and obtain

$$t = x^2 - 1,$$

where $x \geq 1$. Substituting this relation into the equation $y = e^t$, we obtain the Cartesian equation

$$y = e^{x^2-1}.$$ 

Since $y$ is also a one-to-one function of $t$, we have the relation

$$t = \ln y,$$

where $y \geq 1$. We can substitute this relation into the equation $x = \sqrt{t+1}$ to obtain the alternative representation of the curve,

$$x = \sqrt{\ln y + 1}.$$
Summary

- A parametric curve in the \(xy\)-plane is a curve that is described by parametric equations \(x = f(t)\) and \(y = g(t)\), which define the \(x\)- and \(y\)-coordinates of each point on the curve as functions of a parameter \(t\), where \(t\) belongs to an interval \([a, b]\).
- The initial point of the curve is \((f(a), g(a))\), and the terminal point is \((f(b), g(b))\).
- Any curve can be described by parametric equations, because parametric equations do not require that each \(x\)-value is associated with only one \(y\)-value, unlike an equation of the form \(y = f(x)\).
- A curve defined by an equation of the form \([f(x)]^2 + [g(y)]^2 = r^2\) can be converted to parametric equations by equating \(f(x) = r \cos t\) and \(g(y) = r \sin t\), and solving for \(x\) and \(y\).
- Parametric equations \(x = f(t), y = g(t)\) can be converted to an equation of the form \(y = f(x)\) by solving \(x = f(t)\) for \(t\), if possible, and substituting the resulting expression for \(t\) into the equation \(y = g(t)\).