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Lecture 30 Notes

These notes correspond to Section 7.4 in the text.

Arc Length

In this lecture, we will learn how to use calculus to compute the length of a curve that is described by an equation of the form $y = f(x)$, for some given function $f(x)$. Just as we learned how to compute the area under such a curve as the limit of a sum of areas of simpler regions (namely, rectangles), we can compute the length of the curve by interpreting the length as a limit of a sum of lengths of the simplest curves known, which are line segments.

Suppose that we wish to compute the length of the curve $y = f(x)$ between $x = a$ and $x = b$. We can approximate this length by dividing the interval $[a, b]$ into subintervals of length $\Delta x = (b-a)/n$, just as we did when we were trying to compute the approximate area under $y = f(x)$. Consider any subinterval $[x_{i-1}, x_i]$. Then, if Δx is chosen to be sufficiently small, the length of the curve $y = f(x)$ between $x = x_{i-1}$ and $x = x_i$ can be well approximated by the length of the line segment between the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. This line segment is the hypotenuse of a right triangle with legs of length Δx and $|f(x_i) - f(x_{i-1})|$, and therefore the length L_i of the curve $y = f(x)$ between $x = x_{i-1}$ and $x = x_i$ is approximately

$$L_i \approx \sqrt{\Delta x^2 + (f(x_i) - f(x_{i-1}))^2} = \Delta x \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2}. \quad (1)$$

It follows that the length L of the curve between $x = a$ and $x = b$ is approximated by

$$L \approx \sum_{i=1}^n L_i = \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2} \Delta x = \sum_{i=1}^n \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \Delta x. \quad (2)$$

As n , the number of line segments, approaches ∞ , Δx approaches zero, so the length of each subinterval $[x_{i-1}, x_i]$ tends to zero. It follows from the definition of the derivative that

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \lim_{\Delta x \rightarrow 0} \frac{f(x_{i-1} + \Delta x) - f(x_{i-1})}{\Delta x} = f'(x_{i-1}) \quad (3)$$

and therefore the sum converges to the definite integral

$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n L_i = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (4)$$

The value of this integral is called the *arc length* of the curve $y = f(x)$ from $x = a$ to $x = b$. Similarly, if a curve is defined by the equation $x = f(y)$ from $y = c$ to $y = d$, the arc length of the curve is given by the definite integral

$$\int_c^d \sqrt{1 + [f'(y)]^2} dy. \quad (5)$$

Example 1 Compute the arc length of the curve $y = 2x + 3$, where $0 \leq x \leq 2$.

Solution Since the curve is just a line segment, we can simply use the distance formula to compute the arc length, since the arc length is the distance between the endpoints of the segment. The endpoints are $(0, 3)$ and $(2, 7)$, and therefore the arc length is

$$\sqrt{(2 - 0)^2 + (7 - 3)^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}. \quad (6)$$

Using the arc length formula, we have $y' = 2$, and therefore the arc length is given by the integral

$$\int_0^2 \sqrt{1 + 2^2} dx = \int_0^2 \sqrt{5} dx = \sqrt{5}x \Big|_0^2 = 2\sqrt{5}. \quad (7)$$

□

Example 2 Compute the arc length of the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

Solution Since $y' = \cos x$, the arc length is given by the integral

$$\int_0^\pi \sqrt{1 + \cos^2 x} dx. \quad (8)$$

Unfortunately, this integral cannot be evaluated using the Fundamental Theorem of Calculus. Using an approximation method such as the Composite Simpson's Rule, the value of the integral is seen to be approximately 3.8202. □

Example 3 Compute the arc length of the *astroid* described by the equation $x^{2/3} + y^{2/3} = 1$.

Solution We consider only the portion of the astroid in the upper quadrant $x \geq 0, y \geq 0$, which has endpoints $(0, 1)$ and $(1, 0)$. In this quadrant, the astroid can be described by the equation

$$y = (1 - x^{2/3})^{3/2}. \quad (9)$$

It follows that the arc length L of this segment of the astroid is given by the integral

$$L = \int_0^1 \sqrt{1 + (y')^2} dx$$

$$\begin{aligned}
&= \int_0^1 \sqrt{1 + ((3/2)(1 - x^{2/3})^{1/2}(-2/3)x^{-1/3})^2} dx \\
&= \int_0^1 \sqrt{1 + (-(1 - x^{2/3})^{1/2}x^{-1/3})^2} dx \\
&= \int_0^1 \sqrt{1 + (1 - x^{2/3})x^{-2/3}} dx \\
&= \int_0^1 \sqrt{1 + (x^{-2/3} - 1)} dx \\
&= \int_0^1 \sqrt{x^{-2/3}} dx \\
&= \int_0^1 x^{-1/3} dx \\
&= \left. \frac{3}{2}x^{2/3} \right|_0^1 \\
&= \frac{3}{2}.
\end{aligned} \tag{10}$$

□

Example 4 Prove that the shortest distance between two given points is a straight line.

Solution For simplicity, we assume that the two points lie on the same horizontal line; specifically, the points are (a, k) and (b, k) . Let $y = f(x)$ describe a curve connecting the two points. Then, the arc length of the curve is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx. \tag{11}$$

Since the integrand $\sqrt{1 + [f'(x)]^2}$ is always positive, we can minimize the arc length by choosing $f(x)$ so that the integrand itself is minimized. This is the case when $f'(x) = 0$; i.e., $f(x)$ is constant. Therefore the arc length is minimized when $f(x) = k$ and the corresponding curve is a straight line connecting the two points. □

In some cases, it is desirable to compute the arc length of a curve $y = f(x)$ as a function of its endpoints. For example, if the left endpoint of the curve is fixed at the point $(a, f(a))$ and we wish to know the arc length along this curve from the left endpoint to any other point $(x, f(x))$, then we can obtain this length as a function of x from the integral

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt. \tag{12}$$

The function $s(x)$ is known as the *arc length function*.

Summary

- The arc length of a curve defined by the equation $y = f(x)$, for $a \leq x \leq b$, is the integral of $\sqrt{1 + [f'(x)]^2}$ from a to b .
- Often, this integral cannot be computed analytically using known integration rules, so the arc length must instead be approximated numerically using a technique such as the Composite Simpson's Rule.