MAT 280: Multivariable Calculus

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Chapter 1

Partial Derivatives

1.1 Introduction

This course is the fourth course in the calculus sequence, following MAT 167, MAT 168 and MAT 169. Its purpose is to prepare students for more advanced mathematics courses, particularly courses in mathematical programming (MAT 419), advanced engineering mathematics (MAT 430), real analysis (MAT 441), complex analysis (MAT 436), and numerical analysis (MAT 460 and 461). The course will focus on three main areas, which we briefly discuss here.

1.1.1 Partial Differentiation

In single-variable calculus, you learned how to compute the derivative of a function of one variable, $y = f(x)$, with respect to its independent variable x, denoted by dy/dx. In this course, we consider functions of several variables. In most cases, the functions we use will depend on two or three variables, denoted by x, y and z, corresponding to spatial dimensions.

When a function $f(x, y, z)$, for example, depends on several variables, it is not possible to describe its rate of change with respect to those variables using a single quantity such as the derivative. Instead, this rate of change is a *vector* quantity, called the *gradient*, denoted by ∇f. Each component of the gradient is the *partial derivative* of f with respect to one of its independent variables, x, y or z. That is,

$$\nabla f = \left[\begin{array}{ccc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{array} \right].$$

For example, the partial derivative of f with respect to x, denoted by $\partial f/\partial x$, describes the instantaneous rate of change of f with respect to x, when y and z are kept constant. Partial derivatives can be computed using the same differentiation techniques as in single-variable calculus, but one must be careful, when differentiating with respect to one variable, to treat all other variables as if they are constant. For example, if $f(x, y) = x^2y+y^3$, then

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 3y^2,$$

because the y^3 term does not depend on x, and therefore its partial derivative with respect to x is zero.

If

$$\mathbf{F}(x, y, z) = \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{bmatrix}$$

is a *vector-valued* function of three variables, then each of its *component functions* F_1, F_2, and F_3 has a gradient vector, and the rate of change of $\mathbf{F}$ with respect to x, y and z is described by a *matrix*, called the *Jacobian matrix*

$$J_{\mathbf{F}}(x, y, z) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix},$$

where each entry of $J_{\mathbf{F}}(x, y, z)$ is a partial derivative of one of the component functions with respect to one of the independent variables.

We will learn how to generalize various concepts and techniques from single-variable differential calculus to the multi-variable case. These include:

- tangent lines, which become *tangent planes* for functions of two variables and tangent spaces for functions of three or more variables. These are used to compute linear approximations similar to those of functions of a single variable.

- The Chain Rule, which generalizes from a product of derivatives to a product of Jacobian matrices, using standard matrix multiplication. This allows computing the rate of change of a function as its independent variables change along *any* direction in space, not just along any of the coordinate axes, which in turn allows determination of the direction in which a function increases or decreases most rapidly.

- computing maximum and minimum values of functions, which, in the multi-variable case, requires finding points at which the gradient is equal to the zero vector (corresponding to finding points at which the derivative is equal to zero) and checking whether the matrix of second partial derivatives is *positive definite* for a minimum, or *negative definite* for a maximum (which generalizes the second derivative test from the single-variable case). We will also learn how to compute maximum and minimum values subject to constraints on the independent variables, using the method of *Lagrange multipliers*.

1.1.2 Multiple Integration

Next, we will learn how to compute integrals of functions of several variables over multi-dimensional domains, generalizing the definite integral of a function $f(x)$ over an interval $[a, b]$. The integral of a function of two variables $f(x, y)$ represents the volume under a surface described by the graph of f, just as the integral of $f(x)$ is the area under the curve described by the graph of f.

In some cases, it is more convenient to evaluate an integral by first performing a change of variables, as in the single-variable case. For example, when integrating a function of two variables, polar coordinates is useful. For functions of three variables, cylindrical and spherical coordinates, which are both generalizations of polar coordinates, are worth considering.

In the general case, evaluating the integral of a function of n variables by first changing to n different variables requires multiplying the integrand by the determinant of the Jacobian matrix of the function that maps the new variables to the old. This is a generalization of the u-substitution from single-variable calculus, and also relates to formulas for area and volume from MAT 169 that are defined in terms of determinants, or equivalently, in terms of the dot product and cross product.

1.1.3 Vector Calculus

In the last part of the course, we will study *vector fields*, which are functions that assign a vector to each point in its domain, like the vector-valued function $\mathbf{F}$ described above. We will first learn how to compute *line integrals*, which are integrals of functions along curves. A line integral can be viewed as a generalization of the integral of a function on an interval, in that dx is replaced by ds, an infinitesimal distance between points on the curve. It can also be viewed as a generalization of an integral that computes the arc length of a curve, as the line integral of a function that is equal to one yields the arc length. A line integrals of a vector field is useful for computing the work done by a force applied to an object to move it along a curved path. To facilitate the computation of line integrals, a variation of the Fundamental Theorem of Calculus is introduced.

Next, we generalize the notion of a parametric curve to a parametric surface, in which the coordinates of points on the surface depend on two parameters u and v, instead of a single parameter t for a parametric curve. Using the relation between the cross product and the area of a parallelogram, we can define the integral of a function over a parametric surface, which is similar to how a change of variables in a double integral is handled. Then, we will learn how to integrate vector fields over parametric surfaces, which is useful for computing the mass of fluid that crosses a surface, given the rate of flow per unit area.

We conclude with discussion of several fundamental theorems of vector calculus: Green's Theorem, Stokes' Theorem, and the Divergence Theorem. All of these can be seen to be generalizations of the Fundamental Theorem of Calculus to higher dimensions, in that they relate the integral of a function over the interior of a domain to an integral of a related function over its boundary. These theorems can be conveniently stated using the div and curl operations on vector fields. Specifically, if $\mathbf{F} = \langle P, Q, R\rangle$, then

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle.$$

However, using the language of *differential forms*, we can condense the Fundamental Theorem of Calculus and all four of its variations into one theorem, known as the *General Stokes' Theorem*. We now state all six results; their discussion is deferred to Chapter 3.

Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

where f is continuously differentiable on $[a, b]$

Fundamental Theorem of Line Integrals:

$$\int_a^b \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$, is the position function for a curve C and $f(x, y, z)$ is a continuously differentiable function defined on C

Green's Theorem:

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

where D is a 2-D region with piecewise smooth boundary ∂D and P and Q are continuously differentiable on D

Stokes' Theorem:

$$\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \int_{\partial S} \mathbf{F}\cdot\mathbf{T}\,ds$$

where S is a surface in 3-D with unit normal vector $\mathbf{n}$, and piecewise smooth boundary ∂S with unit tangent vector $\mathbf{T}$, and $\mathbf{F}$ is a continuously differentiable vector field

Divergence Theorem:

$$\int_E \operatorname{div}\mathbf{F}\,dV = \int_{\partial E} \mathbf{F}\cdot\mathbf{n}\,dS$$

where E is a solid region in 3-D with boundary surface ∂E, which has outward unit normal vector $\mathbf{n}$, and $\mathbf{F}$ is a continuously differentiable vector field

General Stokes' Theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

where M is a k-manifold and ω is a $(k-1)$-form on M

1.2 Functions of Several Variables

Multi-variable calculus is concerned with the study of quantities that depend on more than one variable, such as temperature that varies within a three-dimensional object, which is a scalar quantity, or the velocity of a flowing liquid, which is a vector quantity. To aid in this study, we first introduce some important terminology and notation that is useful for working with functions of more than one variable, and then introduce some techniques for visualizing such functions.

1.2.1 Terminology and Notation

The following standard notation and terminology is used to define, and discuss, functions of several variables and their visual representations. As they will be used throughout the course, it is important to become acquainted with them immediately.

- The set $\mathbb{R}$ is the set of all real numbers.

- If S is a set and $\mathbf{x}$ is an element of S, we write $\mathbf{x} \in S$.

  Example $2 \in \mathbb{R}$, and $\pi \in \mathbb{R}$, but $i \notin \mathbb{R}$, where $i = \sqrt{-1}$ is an imaginary number.

- If S is a set and T is a subset of S (that is, every element of T is in S), we write $T \subseteq S$.

  Example The set of natural numbers, $\mathbb{N}$, and the set of integers, $\mathbb{Z}$, satisfy $\mathbb{N} \subseteq \mathbb{Z}$. Furthermore, $\mathbb{Z} \subseteq \mathbb{R}$.

- The set $\mathbb{R}^n$ is the set of all ordered n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of real numbers. Each real number x_i is called a *coordinate* of the *point* $\mathbf{x}$.

  Example Each point $(x, y) \in \mathbb{R}^2$ has coordinates x and y. Each point $(x, y, z) \in \mathbb{R}^3$ has an x-, y- and z-coordinate.

- A *function* f with *domain* $D \subseteq \mathbb{R}^n$ and *range* $R \subseteq \mathbb{R}^m$ is a set of ordered pairs of the form $\{(\mathbf{x}, \mathbf{y})\}$, where $\mathbf{x} \in D$ and $\mathbf{y} \in R$, such that each element $\mathbf{x} \in D$ is *mapped* to only one element of R. That is, there is only one ordered pair in f such that $\mathbf{x}$ is its first element. We write $f : D \to R$ to indicate that f maps elements of D to elements of R. We also say that f maps D into R.

  Example Let R^+ denote the set of non-negative real numbers. The function $f(x, y) = x^2 + y^2$ maps $\mathbb{R}^2$ into R^+, and we can write $f : \mathbb{R}^2 \to R^+$.

- Let $f : D \to R$, and let $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}^m$. If $m = 1$, we say that f is a *scalar-valued function*, and if $m > 1$, we say that f is a *vector-valued function.* If $n = 1$, we say that f is a *function of one variable*, and if $n > 1$, we say that f is a *function of several variables.* For each $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D$, the coordinates $x_1, x_2, \ldots, x_n$ of $\mathbf{x}$ are called the *independent variables* of f, and for each $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in R$, the coordinates $y_1, y_2, \ldots, y_m$ of $\mathbf{y}$ are called the *dependent variables* of f.

Example The function $z = x^2 + y^2$ is a scalar-valued function of several variables. The independent variables are x and y, and the dependent variable is z. The function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $x(t) = t\cos t$, $y(t) = t\sin t$, and $z(t) = e^t$, is a vector-valued function with independent variable t and dependent variables x, y and z.

- Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. The *graph* of f is the subset of $\mathbb{R}^{n+1}$ consisting of the points $(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$, where $(x_1, x_2, \ldots, x_n) \in D$.

  Solution The graph of the function $z = x^2 + y^2$ is a *parabola of revolution* obtained by revolving the parabola $z = x^2$ around the z-axis. The graph of the function $z = x + y - 1$ is a plane in 3-D space that passes through the points $(0, 0, -1)$ and $(1, 1, 1)$.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *linear function* if f has the form

  $$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + b,$$

  where $a_1, a_2, \ldots, a_n$ and b are constants.

  Example The function $y = mx + b$ is a linear function of the single independent variable x. Its graph is a line contained within the xy-plane, with slope m, passing through the point $(0, b)$. The function $z = ax + by + c$ is a linear function of the two independent variables x and y. Its graph is a plane in 3-D space that passes through the points $(0, 0, c)$ and $(1, 1, a + b + c)$.

- Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that a set L is a *level set* of f if $L \subseteq D$ and f is equal to a constant value k on L; that is, $f(\mathbf{x}) = k$ if $\mathbf{x} \in L$. If $n = 2$, we say that L is a *level curve* or *level contour*; if $n = 3$, we say that L is a *level surface*.

  Example A level surface of the function $f(x, y, z) = x^2+y^2+z^2$, where $f(x, y, z) = k$ for a constant k, is a sphere of radius $\sqrt{k}$. The level curves of the function $z = x^2 + y^2$ are circles of radius $\sqrt{k}$ with center $(0, 0, k)$, situated in the plane $z = k$, for each nonnegative number k.

1.2.2 Visualization Techniques

While it is always possible to obtain the graph of a function $f(x, y)$, for example, by substituting various values for its independent variables and plotting the corresponding points from the graph, this approach is not necessarily helpful for understanding the graph as a whole. Knowing the extent of the possible values of a function's independent and dependent variables (the domain and range, respectively), along with the behavior of a few select curves that are contained within the function's graph, can be more helpful. To that end, we mention the following useful techniques for acquiring this information.

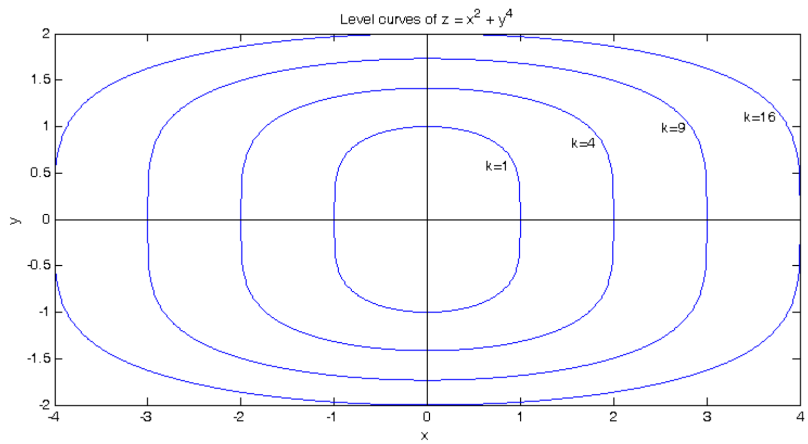

Figure 1.1: Level curves of the function $z = x^2 + y^4$

- To find the domain and range of a function f, it is often necessary to account for the domains of functions that are included in the definition of f. For example, if there is a square root, it is necessary to avoid taking the square root of a negative number.

  Example Let $f(x, y) = \ln(x^2 - y^2)$. Since $\ln |x|$ is only defined for $x > 0$, we must have $x^2 > y^2$, which, upon taking the square root of both sides, yields $|x| > |y|$. Therefore, this inequality defines the domain of f. The range of f is the range of ln, which is $\mathbb{R}$.

- To find the level set of a function $f(x_1, x_2, \ldots, x_n)$, solve the equation $f(x_1, x_2, \ldots, x_n) = k$, where k is a constant. This equation will implicitly define the level set. In some cases, it can be solved for one of

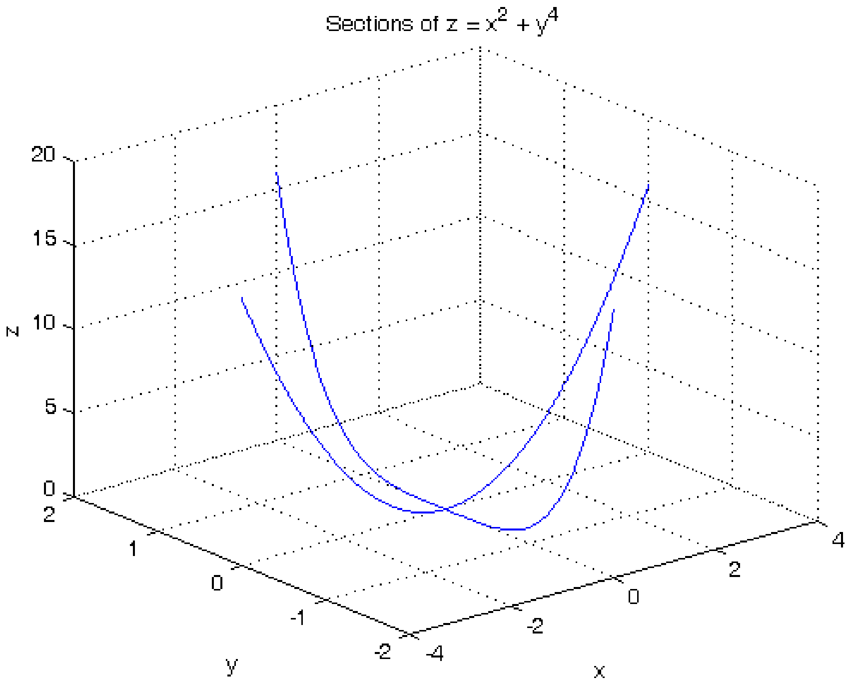

Figure 1.2: Sections of the function $z = x^2 + y^4$

the independent variables to obtain an explicit function that describes the level set.

Example Let $z = 2x + y$ be a function of two variables. The graph of this function is a plane. Each level set of this function is described by an equation of the form $2x + y = k$, where k is a constant. Since $z = k$ as well, the graph of this level set is the line with equation $y = -2x + k$, contained within the plane $z = k$.

Example Let $z = \ln y - x$. Each level set of this function is described by an equation of the form $\ln y - x = k$, where k is a constant. Exponentiating both sides, we obtain $y = e^{x+k}$. It follows that the graph of this level set is that of the exponential function, contained within the plane $z = k$, and shifted k units to the left (that is, along the x-axis).

- To help visualize a function of two variables $z = f(x, y)$, it can be helpful to use the *method of sections*. This involves viewing the functions when restricted to "vertical" planes, such as the xz-plane and the yz-plane. To take these two sections, first set $y = 0$ to obtain z as a function of x, and then graph that function in the xz-plane. Then, set $x = 0$ to obtain z as a function of y, and graph that function in the yz-plane. Using these graphs as guides, in conjunction with level curves, it is then easier to visualize what the rest of the graph of f looks like.

Example Let $z = x^2 + y^4$. Setting $y = 0$ yields $z = x^2$, the graph of which is a parabola in the xz-plane. Setting $x = 0$ yields $z = y^4$, which has a graph that is a parabola-like curve, where z increases much more rapidly. Combining these graphs with selected level curves, which are described by the equations $y = \sqrt[4]{k - x^2}$, where $|x| \leq \sqrt{k}$ for $k \geq 0$, allows us to visualize the graph of this function. Level curves and sections are shown in Figures 1.1 and 1.2, respectively.

1.3 Limits and Continuity

Recall that in single-variable calculus, the fundamental concept of a limit was used to define derivatives and integrals of functions, as well as the notion of continuity of a function. We now generalize limits and continuity to the case of functions of several variables.

1.3.1 Terminology and Notation

- Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$, and $a \in D$. We say $f(x)$ *approaches* L *as* x *approaches* a, and write

$$\lim_{x \to a} f(x) = L$$

  if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

- If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a point in $\mathbb{R}^n$, or, equivalently, if $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ is a position vector in $\mathbb{R}^n$, then the *magnitude*, or *length*, of $\mathbf{x}$, denoted by $\|\mathbf{x}\|$, is defined by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}.$$

  Note that if $n = 1$, then $\mathbf{x}$ is actually a scalar x, and $\|\mathbf{x}\| = |x|$.

Example If $\mathbf{x} = (3, -1, 4) \in \mathbb{R}^3$, then $\|\mathbf{x}\| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26}$. □

- Let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and $\mathbf{a} \in D$. We say $\mathbf{f}(\mathbf{x})$ *approaches* $\mathbf{b}$ *as* $\mathbf{x}$ *approaches* $\mathbf{a}$, and write

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b},$$

if, for any $\epsilon > 0$, *no matter how small*, there exists a $\delta > 0$ such that for *any* $\mathbf{x}$ such that $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, $\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| < \epsilon$. This definition is illustrated in Figure 1.3. Note that the condition $\|\mathbf{x} - \mathbf{a}\| > 0$ specifically excludes consideration of $\mathbf{x} = \mathbf{a}$, because limits are used to understand the behavior of a function *near* a point, not *at* a point.

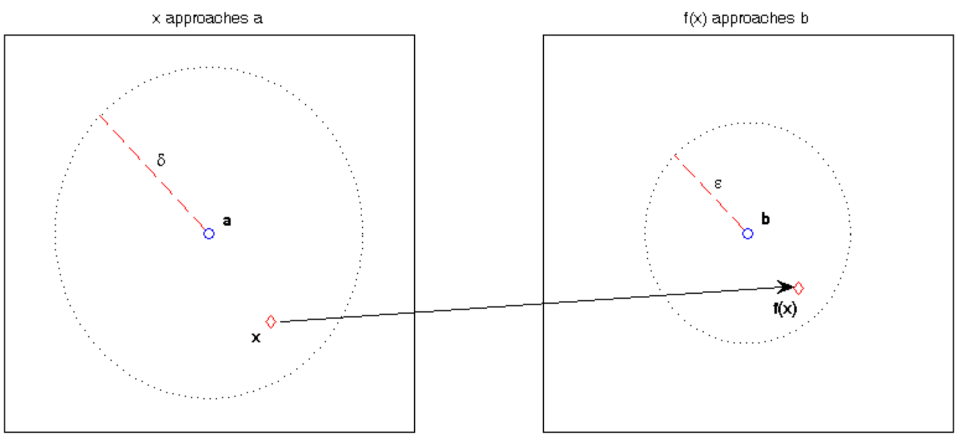

Figure 1.3: Illustration of the limit, as $\mathbf{x}$ approaches $\mathbf{a}$ (left plot), of $f(\mathbf{x})$ being equal to $\mathbf{b}$ (right plot). For any ball around the point $\mathbf{b}$ of radius ϵ (right plot), *no matter how small*, there *exists* a ball around the point $\mathbf{a}$, of radius δ (left plot), such that *every* point in the ball around $\mathbf{a}$ is mapped by f to a point in the ball around $\mathbf{b}$.

Example Consider the function

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}.$$

We will use the definition of a limit to show that as $(x, y) \to (0, 0)$, $f(x, y) \to 0$. Let $\epsilon > 0$. We need to show that there exists some $\delta > 0$

such that if $0 < \|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$, then $|f(x,y) - 0| = |\frac{xy}{\sqrt{x^2+y^2}}| < \epsilon$. First, we note that

$$\left|\frac{y}{\sqrt{x^2+y^2}}\right| < \left|\frac{y}{\sqrt{y^2}}\right| = \left|\frac{y}{|y|}\right| = 1.$$

Therefore, if we set $\delta = \epsilon$, we obtain

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = |x|\left|\frac{y}{\sqrt{x^2+y^2}}\right| < |x| = \sqrt{x^2} < \sqrt{x^2+y^2} < \delta = \epsilon,$$

from which it follows that the limit exists and is equal to zero. □

- Let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and let $\mathbf{a} \in D$. We say that f is *continuous* at $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$.

- Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that f is a *polynomial* if, for each $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D$, $f(\mathbf{x})$ is equal to a sum of terms of the form $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$, where $p_1, p_2, \ldots, p_n$ are nonnegative integers.

  Example The functions $f(x) = x^3+3x^2+2x+1$, $g(x,y) = x^2y^3+3xy+x^2+1$, and $h(x,y,z) = 4xy^2z^3+8yz^2$ are all examples of polynomials. □

- Let $f, p, q : D \subseteq \mathbb{R}^n \to \mathbb{R}$, and let $q(\mathbf{x}) \neq 0$ on D. We say that f is a *rational function* if p and q are both polynomials and $f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x})$.

  Example The functions $f(x) = 1/(x+1)$, $g(x,y) = xy^2/(x^2+y^3)$, and $h(x,y,z) = (xy^2+z^3)/(x^2z+xyz^2+yz^3)$ are all examples of rational functions. □

- An *algebraic* function is a function that satisfies a polynomial equation whose coefficients are themselves polynomials.

  Example The square root function $y = \sqrt{x}$ is an algebraic function, because it satisfies the equation $y^2 - x = 0$. The function $y = x^{5/2} + x^{3/2}$ is also an algebraic function, because it satisfies the equation $x^5 + 2x^4 + x^3 - y^2 = 0$. □

1.3.2 Defining Limits Using Neighborhoods

An alternative approach to defining limits involves the concept of a *neighborhood*, which generalizes open intervals on the real number line.

- Let $\mathbf{x}_0 \in \mathbb{R}^n$ and let $r > 0$. We define the *ball centered at* $\mathbf{x}_0$ *of radius* r, denoted by $D_r(\mathbf{x}_0)$, to be the set of all points $\mathbf{x} \in \mathbb{R}^n$ such that $\|\mathbf{x} - \mathbf{x}_0\| < r$.

  Example In 1-D, the open interval $(0, 1)$ is also the ball centered at $x_0 = 1/2$ of radius $r = 1/2$. In 3-D, the inside of the sphere with center $(0, 0, 0)$ and radius 2, $\{(x, y, z)|x^2 + y^2 + z^2 < 4\}$, is also the ball $D_2((0, 0, 0))$. □

- We say that a set $U \subseteq \mathbb{R}^n$ is *open* if, for any point $\mathbf{x}_0 \in U$, there exists an $r > 0$ such that $D_r(\mathbf{x}_0) \subseteq U$.

  Example In 1-D, any open set is an open interval, such as $(-1, 1)$, or a union of open intervals. In 2-D, the interior of the ellipse defined by the equation $4x^2 + 9y^2 = 1$ is an open set; the ellipse itself is *not* included. □

- Let $\mathbf{x}_0 \in \mathbb{R}^n$. We say that N is a *neighborhood* of $\mathbf{x}_0$ if N is an open set that contains $\mathbf{x}_0$.

- Let $A \subseteq \mathbb{R}^n$ be an open set. We say that $\mathbf{x}_0 \in \mathbb{R}^n$ is a *boundary point* of A if *every* neighborhood of $\mathbf{x}_0$ contains at least one point in A and one point not in A.

  Example Let $D = \{(x, y)|x^2 + y^2 < 1\}$, which is often called the *unit ball* in $\mathbb{R}^2$. This set consists of all points inside the unit circle with center $(0, 0)$ and radius 1, not including the circle itself. The point $(x_0, y_0) = (\sqrt{2}/2, \sqrt{2}/2)$, which is on the circle, is a boundary point of D because, as illustrated in Figure 1.4, any neighborhood of (x_0, y_0) must contain points inside the circle, and points that are outside. □

- Let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and let $\mathbf{a} \in D$ or let $\mathbf{a}$ be a boundary point of D. We say that $\mathbf{a} \in D$. We say that $\mathbf{f}(\mathbf{x})$ approaches $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{a}$, and write

  $$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b},$$

  if, for any neighborhood N of $\mathbf{b}$, there exists a neighborhood U of $\mathbf{a}$ such that if $\mathbf{x} \in U$, then $\mathbf{f}(\mathbf{x}) \in N$.

1.3.3 Results

In the statement of the following results concerning limits and continuity, $\mathbf{f}, \mathbf{g} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{a} \in D$ or $\mathbf{a}$ is a boundary point of D, $\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^m$, and $c \in \mathbb{R}$.

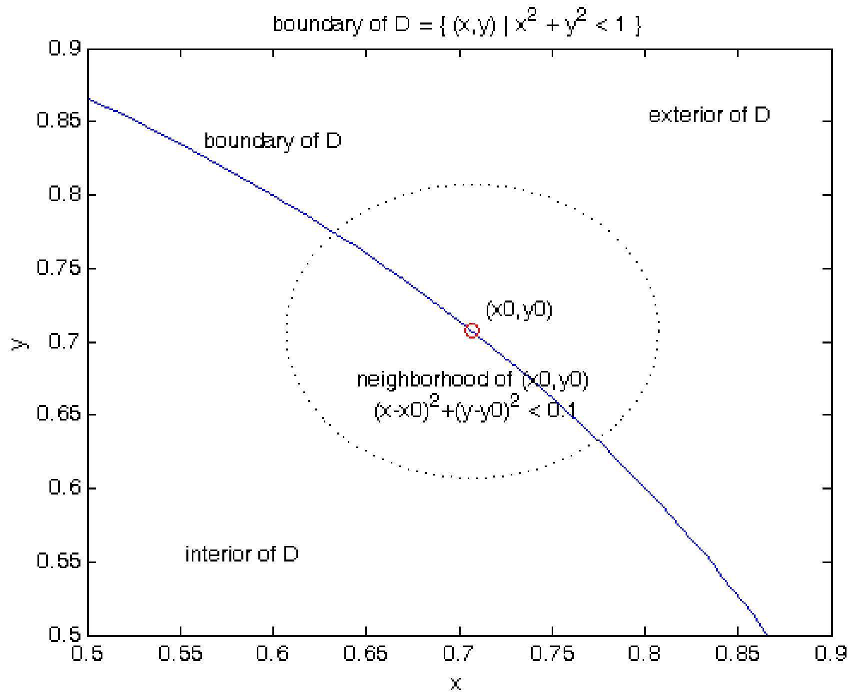

Figure 1.4: Boundary point (x_0, y_0) of the set $D = \{(x, y)|x^2 + y^2 < 1\}$. The neighborhood of (x_0, y_0) shown, $D_r((x_0, y_0)) = \{(x, y)|(x - x_0)^2 + (y - y_0)^2 < 0.1\}$, contains points that are in D and points that are not in D.

,

- The limit of a function $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$, if it exists, is unique. That is, if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}_1 \quad \text{and} \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}_2,$$

then $\mathbf{b}_1 = \mathbf{b}_2$. It follows that if $\mathbf{f}(\mathbf{x})$ approaches two *distinct* values as $\mathbf{x}$ approaches $\mathbf{a}$ along two *distinct* paths, then the limit as $\mathbf{x}$ approaches $\mathbf{a}$ does *not* exist.

- If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x}\to\mathbf{a}} c\mathbf{f}(\mathbf{x}) = c\mathbf{b}$.

- If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{b}_2$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f} + \mathbf{g})(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2.$$

Furthermore, if $m = 1$, then

$$\lim_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = b_1 b_2.$$

- If $m = 1$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b \neq 0$, and $f(\mathbf{x}) \neq 0$ in a neighborhood of $\mathbf{a}$, then
$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{1}{f(\mathbf{x})} = \frac{1}{b}.$$

- If $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $f_1, f_2, \ldots, f_m$ are the component functions of $\mathbf{f}$, and $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$, then $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ if and only if $\lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = b_i$ for $i = 1, 2, \ldots, m$.

- If $\mathbf{f}$ and $\mathbf{g}$ are continuous at $\mathbf{a}$, then so is $c\mathbf{f}$ and $f + g$. If, in addition, $m = 1$, then fg is continuous at $\mathbf{a}$. Furthermore, if $m = 1$ and if f is nonzero in a neighborhood of $\mathbf{a}$, then $1/f$ is continuous at $\mathbf{a}$.

- If $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $f_1, f_2, \ldots, f_m$ are the component functions of $\mathbf{f}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$ if and only if f_i is continuous at $\mathbf{a}$, for $i = 1, 2, \ldots, m$.

- Any polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on all of $\mathbb{R}^n$.

- Any rational function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous wherever it is defined.

  Example The function $f(x, y) = 2x/(x^2 - y^2)$ is defined on all of $\mathbb{R}^2$ except where $x^2 - y^2 = 0$; that is, where $|x| = |y|$. Therfore, f is continuous at all such points. □

- Let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and let $\mathbf{g} : U \subseteq \mathbb{R}^p \to D$. If the *composition* $(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$ defined on U, then $\mathbf{f} \circ \mathbf{g}$ is continuous at $\mathbf{a} \in U$ if $\mathbf{g}$ is continuous at $\mathbf{a}$ and $\mathbf{f}$ is continuous at $\mathbf{g}(\mathbf{a})$.

  Example The function $g(x, y) = x^2 + y^2$, being a polynomial, is continuous on all of $\mathbb{R}^2$. The function $f(z) = \sin z$ is continuous on all of $\mathbb{R}$. Therefore, the composition $(f \circ g)(x, y) = f(g(x, y)) = \sin(x^2 + y^2)$ is continuous on all of $\mathbb{R}^2$. □

- Algebraic functions, such as x^r where r is any rational number (for example, $f(x) = \sqrt{x}$) and trigonometric functions, such as $\sin x$ or $\tan x$, are continuous wherever they are defined.

1.3.4 Techniques for Establishing Limits and Continuity

We now discuss some techniques for computing limits of functions of several variables, or determining that they do not exist. We also demonstrate how to determine if a function is continuous at a point.

To show that the limit of a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ as $\mathbf{x} \to \mathbf{a}$ does *not* exist, try letting $\mathbf{x}$ approach $\mathbf{a}$ along different paths to see if different values are approached. If they are, then the limit does not exist.

For example, let $n = 2$ and let $\mathbf{x} = (x, y)$ and $\mathbf{a} = (a_1, a_2)$. Then, try setting $x = a_1$ in the formula for $f(x, y)$ and letting y approach a_2, or vice versa. Other possible paths include, for example, setting $x = cy$, where $c = a_1/a_2$, if $a_2 \neq 0$, and letting y approach a_2, or considering the cases of $x < a_1$ and $x > a_1$, or $y < a_2$ and $y > a_2$, separately.

Example Let $f(x, y) = x^3y/(x^4+y^4)$. If we let $(x, y) \to (0, 0)$ by first setting $y = 0$ and then letting $x \to 0$, we observe that $f(x, 0) = x^3(0)/(x^4 + 0) = 0$ for all $x \neq 0$. This suggests that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$. However, if we set $x = y$ and let $x, y \to 0$ together, we note that $f(x, x) = x^3x/(x^4 + x^4) = x^4/(2x^4) = 1/2$, which suggests that the limit is equal to $1/2$. We conclude that the limit does not exist. $\Box$

To show that the limit of a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ as $\mathbf{x} \to \mathbf{a}$ *does* exist and is equal to b, use the definition of a limit by first assuming $\epsilon > 0$, and then trying to find a δ so that $|f(\mathbf{x}) - b| < \epsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$.

To that end, try to find an *upper bound* on $|f(\mathbf{x}) - b|$ in terms of $\|\mathbf{x} - \mathbf{a}\|$. Specifically, if it can be shown that $|f(\mathbf{x}) - b| < g(\|\mathbf{x} - \mathbf{a}\|)$, where g is an invertible, increasing function, then a suitable choice for δ is $\delta = g^{-1}(\epsilon)$. Then, if $\|\mathbf{x} - \mathbf{a}\| < \delta = g^{-1}(\epsilon)$, then

$$|f(\mathbf{x}) - b| < g(\|\mathbf{x} - \mathbf{a}\|) < g(g^{-1}(\epsilon)) = \epsilon.$$

Example Let $f(x, y) = (x^3 - y^3)/(x^2 + y^2)$. Letting $(x, y) \to (0, 0)$ along various paths, it appears that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$. To confirm this, we assume $\epsilon > 0$ and try to find $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then $|(x^3 - y^3)/(x^2 + y^2)| < \epsilon$.

Factoring the numerator of $f(x, y)$, we obtain

$$\left|\frac{x^3 - y^3}{x^2 + y^2}\right| = \left|\frac{(x - y)(x^2 + xy + y^2)}{x^2 + y^2}\right| = \left|(x - y)\left(1 + \frac{xy}{x^2 + y^2}\right)\right|.$$

Using $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$, and similarly $|y| \leq \sqrt{x^2+y^2}$, yields

$$\begin{aligned}\left|\frac{x^3-y^3}{x^2+y^2}\right| &= |x-y|\left|1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right| \\ &\leq 2|x-y| \\ &\leq 2(|x|+|y|) \\ &\leq 4\sqrt{x^2+y^2}.\end{aligned}$$

Therefore, if we let $\delta = \epsilon/4$, it follows that when $\sqrt{x^2+y^2} < \delta$, then $|f(x,y)| < 4\delta = 4(\epsilon/4) = \epsilon$, and therefore the limit exists and is equal to zero. □

Sometimes, it is helpful to simplify the formula for a function before attempting to determine its limit.

Example Consider the function

$$f(x,y) = \frac{(x+y)^2-(x-y)^2}{xy}, \quad x,y \neq 0.$$

Expanding the numerator yields, for $x, y \neq 0$,

$$f(x,y) = \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{xy} = \frac{4xy}{xy} = 4.$$

Therefore, even though $f(x,y)$ is not defined at $(0,0)$, its limit as $(x,y) \to (0,0)$ exists, and is equal to 4. This example demonstrates that a limit depends only on the behavior of a function *near* a particular point; what happens *at* that point is *irrelevant.* □

In many cases, determining whether a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous can be accomplished by applying the various properties of continuous functions stated above, and using the fact that various types of functions, such as polynomial and rational functions, are known to be continuous wherever they are defined.

Example Let $\mathbf{c} = \langle 2, -1, 3\rangle$, and let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathbf{f}(\mathbf{x}) = \mathbf{c}\times\mathbf{x}$, the *cross product* of the vector $\mathbf{c}$ and the vector $\mathbf{x} = \langle x_1, x_2, x_3\rangle$. This function is continuous on all of $\mathbb{R}^3$, because

$$\mathbf{f}(\mathbf{x}) = \mathbf{c}\times\mathbf{x} = \langle -x_3 - 3x_2, 3x_1 - 2x_3, 2x_2 + x1\rangle,$$

and each component function of $\mathbf{f}$ can be seen to be not only a polynomial, but a linear function. □

However, in cases where a function is defined in a piecewise manner, continuity at boundaries between pieces must be determined by applying the definition of continuity directly, which requires computing limits.

Example Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{2x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

For $(x,y) \neq (0,0)$, f is continuous at (x,y) because it is a rational function that is defined. As $(x,y) \to (0,0)$, $f(x,y) \to 0$, as can be shown by applying the definition of a limit with $\delta = \epsilon$. Because this limit is equal to $f(0,0) = 0$, we conclude that f is continuous at $(0,0)$ as well. □

1.4 Partial Derivatives

Now that we have become acquainted with functions of several variables, and what it means for such functions to have limits and be continuous, we are ready to analyze their behavior by computing their instantaneous rates of change, as we know how to do for functions of a single variable. However, in contrast to the single-variable case, the instantaneous rate of change of a function of several variables cannot be described by a single number that represents the slope of a tangent line. Instead, such a slope can only describe how *one* of the function's dependent variables (outputs) varies as *one* of its independent variables (inputs) changes. This leads to the concept of what is known as a *partial derivative*.

1.4.1 Terminology and Notation

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a scalar-valued function of a single variable. Recall that the *derivative of* $f(x)$ *with respect to* x at x_0 is defined to be

$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Now, let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables, and let $(x_0, y_0) \in D$. The *partial derivative of* $f(x,y)$ *with respect to* x at (x_0, y_0) is defined to be

$$\frac{\partial f}{\partial x}(x_0,y_0) = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}.$$

Note that only values of $f(x,y)$ for which $y = y_0$ influence the value of the partial derivative with respect to x. Similarly, the *partial derivative of* $f(x,y)$ *with respect to* y at (x_0,y_0) is defined to be

$$\frac{\partial f}{\partial y}(x_0,y_0) = f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h}.$$

Note the two methods of denoting partial derivatives used above: $\partial f/\partial x$ or f_x for the partial derivative with respect to x. There are other notations, but these are the ones that we will use.

Example Let $f(x,y) = x^2y$, and let $(x_0,y_0) = (2,-1)$. Then

$$\begin{aligned} f_x(2,-1) &= \lim_{h\to 0} \frac{(2+h)^2(-1) - 2^2(-1)}{h} \\ &= \lim_{h\to 0} \frac{-(4+4h+h^2)+4}{h} \\ &= \lim_{h\to 0} \frac{-4h-h^2}{h} \\ &= -4, \\ f_y(2,-1) &= \lim_{h\to 0} \frac{2^2(-1+h) - 2^2(-1)}{h} \\ &= \lim_{h\to 0} \frac{4(h-1)+4}{h} \\ &= \lim_{h\to 0} \frac{4h}{h} \\ &= 4. \end{aligned}$$

□

In the preceding example, the value $f_x(2,-1) = -4$ can be interpreted as the slope of the line that is tangent to the graph of $f(x,-1) = -x^2$ at $x = 2$. That is, we consider the *restriction* of f to the portion of its domain where $y = -1$, and thus obtain a function of the single variable x, $g(x) = f(x,-1) = -x^2$. Note that if we apply differentiation rules from single-variable calculus to g, we obtain $g'(x) = -2x$, and $g'(2) = -4$, which is the value we obtained for $f_x(2,-1)$.

Similarly, if we consider $f_y(2,-1) = 4$, this can be interpreted as the slope of a line that is tangent to the graph of $p(y) = f(2,y) = 4y$ at $y = -1$. Note that if we differentiate p, we obtain $p'(y) = 4$, which, again, shows that the partial derivative of a function of several variables can be obtained by "freezing" the values of all variables except the one with respect to which we are differentiating, and then applying differentiation rules to the resulting function of one variable.

It follows from this relationship between partial derivatives of a function of several variables and the derivative of a function of a single variable that other interpretations of the derivative are also applicable to partial derivatives. In particular, if $f_x(x_0, y_0) > 0$, which is equivalent to $g'(x_0) > 0$ where $g(x) = f(x, y_0)$, we can conclude that f is *increasing* as x varies from x_0, along the line $y = y_0$. Similarly, if $f_y(x_0, y_0) < 0$, which is equivalent to $p'(y_0) < 0$ where $p(y) = f(x_0, y)$, we can conclude that f is *decreasing* as y varies from y_0 along the line $x = x_0$.

We now define partial derivatives for a function of n variables. Let the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ be defined as follows: for each $i = 1, 2, \ldots, n$, $\mathbf{e}_i$ has components that are all equal to zero, except the ith component, which is equal to 1. Then these vectors are called the *standard basis vectors* of $\mathbb{R}^n$.

Example If $n = 3$, then

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We also have that $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$. □

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of n variables $x_1, x_2, \ldots, x_n$. Then, the *partial derivative* of f with respect to x_i at $\mathbf{x}_0 \in \mathbb{R}^n$, where $1 \leq i \leq n$, is defined to be

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) &= f_{x_i}(\mathbf{x}_0) \\ &= \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}. \end{aligned}$$

Example Let $f : \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(\mathbf{x}) = (\mathbf{c} \cdot \mathbf{x})^2$, where $\mathbf{c} \in \mathbb{R}^4$ is the vector $\mathbf{c} = \langle 4, -3, 2, -1 \rangle$. Let $\mathbf{x}_0 \in \mathbb{R}^4$ be the point $\mathbf{x}_0 = \langle 1, 3, 2, 4 \rangle$. Then, the partial derivative of f with respect to x_2 at $\mathbf{x}_0$ is given by

$$\begin{aligned} f_{x_2}(\mathbf{x}_0) &= \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_2) - f(\mathbf{x}_0)}{h} \\ &= \lim_{h \to 0} \frac{(\mathbf{c} \cdot (\mathbf{x}_0 + h\mathbf{e}_2))^2 - (\mathbf{c} \cdot \mathbf{x}_0)^2}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(\mathbf{c}\cdot\mathbf{x}_0 + h\mathbf{c}\cdot\mathbf{e}_2)^2 - (\mathbf{c}\cdot\mathbf{x}_0)^2}{h} \\
&= \lim_{h\to 0} \frac{(\mathbf{c}\cdot\mathbf{x}_0)^2 + 2(\mathbf{c}\cdot\mathbf{x}_0)(h\mathbf{c}\cdot\mathbf{e}_2) + (h\mathbf{c}\cdot\mathbf{e}_2)^2 - (\mathbf{c}\cdot\mathbf{x}_0)^2}{h} \\
&= \lim_{h\to 0} \frac{2h(\mathbf{c}\cdot\mathbf{x}_0)(\mathbf{c}\cdot\mathbf{e}_2) + h^2(\mathbf{c}\cdot\mathbf{e}_2)^2}{h} \\
&= \lim_{h\to 0} 2(\mathbf{c}\cdot\mathbf{x}_0)(\mathbf{c}\cdot\mathbf{e}_2) + h(\mathbf{c}\cdot\mathbf{e}_2)^2 \\
&= 2(\mathbf{c}\cdot\mathbf{x}_0)(\mathbf{c}\cdot\mathbf{e}_2) \\
&= 2(\langle 4,-3,2,-1\rangle\cdot\langle 1,3,2,4\rangle)(\langle 4,-3,2,-1\rangle\cdot\langle 0,1,0,0\rangle) \\
&= 2[4(1) - 3(3) + 2(2) - 1(4)](-3) \\
&= 2(-5)(-3) \\
&= 30.
\end{aligned}
$$

This shows that f is increasing sharply as a function of x_2 at the point $\mathbf{x}_0$. Note that the same result can be obtained by defining

$$
\begin{aligned}
g(x_2) &= f(1, x_2, 2, 4) \\
&= (\mathbf{c}\cdot\langle 1, x_2, 2, 4\rangle)^2 \\
&= (\langle 4,-3,2,-1\rangle\cdot\langle 1, x_2, 2, 4\rangle)^2 \\
&= (4 - 3x_2)^2,
\end{aligned}
$$

differentiating this function of x_2 to obtain $g'(x_2) = 2(4-3x_2)(-3)$, and then evaluating this derivative at $x_2 = 3$ to obtain $g'(3) = 2(4-3(3))(-3) = 30$. □

Just as functions of a single variable can have second derivatives, third derivatives, or derivatives of any order, functions of several variables can have higher-order partial derivatives. To that end, let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of n variables $x_1, x_2, \ldots, x_n$. Then, the *second partial derivative of f with respect to x_i and x_j* at $\mathbf{x}_0 \in D$ is defined to be

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) &= f_{x_i x_j}(\mathbf{x}_0) \\
&= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(\mathbf{x}_0) \\
&= \lim_{h_i\to 0} \frac{f_{x_j}(\mathbf{x}_0 + h_i\mathbf{e}_i) - f_{x_j}(\mathbf{x}_0)}{h_i} \\
&= \lim_{(h_i,h_j)\to(0,0)} \frac{1}{h_i h_j}[f(\mathbf{x}_0 + h_i\mathbf{e}_i + h_j\mathbf{e}_j) - f(\mathbf{x}_0 + h_i\mathbf{e}_i) -
\end{aligned}
$$

$$f(\mathbf{x}_0 + h_j\mathbf{e}_j) + f(\mathbf{x}_0)].$$

The second line of the above definition is the most helpful, in terms of describing how to compute a second partial derivative with respect to x_i and x_j: first, compute the partial derivative with respect to x_j. Then, compute the partial derivative of the result with respect to x_i, and finally, evaluate at the point $\mathbf{x}_0$. That is, the second partial derivative, or a partial derivative of higher order, can be viewed as an *iterated* partial derivative.

A commonly used method of indicating that a function is evaluated at a given point, especially if the formula for the function is complicated or otherwise does not lend itself naturally to the usual notation for evaluation at a point, is to follow the function with a vertical bar, and indicate the evaluation point as a subscript to the bar. For example, given a function $f(x)$, we can write

$$f'(4) = \left.\frac{df}{dx}\right|_{x=4} = \left.\frac{df}{dx}\right|_4$$

or, given a function $f(x, y)$, we can write

$$f_x(2,3) = \left.\frac{\partial f}{\partial x}\right|_{x=2,y=3} = \left.\frac{\partial f}{\partial x}\right|_{(2,3)}.$$

This notation is similar to the use of the vertical bar in the evaluation of definite integrals, to indicate that an antiderivative is to be evaluated at the limits of integration.

1.4.2 Clairaut's Theorem

The following theorem is very useful for reducing the amount of work necessary to compute all of the higher-order partial derivatives of a function.

Theorem (Clairaut's Theorem): Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, and let $\mathbf{x}_0 \in D$. If the second partial derivatives f_{xy} and f_{yx} are continuous on D, then they are equal:

$$f_{xy}(\mathbf{x}_0) = f_{yx}(\mathbf{x}_0).$$

Example Let $f(x, y) = \sin(2x)\cos^2(4y)$. Then

$$f_x = 2\cos^2(4y)\cos(2x), \quad f_y = -8\sin(2x)\cos(4y)\sin(4y),$$

which yields

$$f_{xy} = (2\cos^2(4y)\cos(2x))_y = -16\cos(2x)\cos(4y)\sin(4y)$$

and

$$f_{yx} = (-8\sin(2x)\cos(4y)\sin(4y))_x = -16\cos(2x)\cos(4y)\sin(4y),$$

and we conclude that these *mixed partial derivatives* are equal. □

1.4.3 Techniques

We now describe the most practical techniques for computing partial derivatives. As mentioned previously, computing the partial derivative of a function with respect to a given variable, at a given point, is equivalent to "freezing" the values of all *other* variables at that point, and then computing the derivative of the resulting function of one variable at that point.

However, generally it is most practical to compute the partial derivative as a *function* of all of the independent variables, which can then be evaluated at any point at which we wish to know the *value* of the partial derivative, just as when we have a function $f(x)$, we normally compute its derivative as a function $f'(x)$, and then evaluate that function at any point x_0 where we want to know the rate of change.

Therefore, the most practical approach to computing a partial derivative of a function f with respect to x_i is to apply differentiation rules from single-variable calculus to differentiate f with respect to x_i, while *treating all other variables as constants*. The result of this process is a *function* that represents $\partial f/\partial x_i(x_1, x_2, \ldots, x_n)$, and then values can be substituted for the independent variables $x_1, x_2, \ldots, x_n$.

Example To compute $f_x(\pi/2, \pi)$ of $f(x,y) = e^{-(x^2+y^2)}\sin 3x\cos 4y$, we treat y as a constant, since we are differentiating with respect to x. Using the Product Rule and the Chain Rule from single-variable calculus, as well as the rules for differentiating exponential and trigonometric functions, we obtain

$$\begin{aligned}
f_x(\pi/2,\pi) &= \left.\frac{\partial}{\partial x}[e^{-(x^2+y^2)}\sin 3x\cos 4y]\right|_{x=\pi/2,y=\pi} \\
&= \left.\cos 4y\frac{\partial}{\partial x}[e^{-(x^2+y^2)}\sin 3x]\right|_{x=\pi/2,y=\pi} \\
&= \left.\cos 4y\left[\sin 3x\frac{\partial}{\partial x}[e^{-(x^2+y^2)}] + e^{-(x^2+y^2)}\frac{\partial}{\partial x}[\sin 3x]\right]\right|_{x=\pi/2,y=\pi} \\
&= \cos 4y\left[e^{-(x^2+y^2)}\sin 3x\frac{\partial}{\partial x}[-(x^2+y^2)]+\right.
\end{aligned}$$

$$\begin{aligned}
&3e^{-(x^2+y^2)}\cos 3x\Big]\Big|_{x=\pi/2,y=\pi} \\
&= \cos 4y\left[-2xe^{-(x^2+y^2)}\sin 3x + 3e^{-(x^2+y^2)}\cos 3x\right]\Big|_{x=\pi/2,y=\pi} \\
&= \cos 4\pi\Big[-2(\pi/2)e^{-((\pi/2)^2+\pi^2)}\sin(3\pi/2)+ \\
&\quad 3e^{-((\pi/2)^2+\pi^2)}\cos 3(\pi/2)\Big] \\
&= \pi e^{-5\pi^2/4}.
\end{aligned}$$

Similarly, to compute $f_y(\pi/2,\pi)$, we treat x as a constant, and apply these differentiation rules to differentiate with respect to y. Finally, we substitute $x = \pi/2$ and $y = \pi$ into the resulting derivative. □

This approach to differentiation can also be applied to compute higher-order partial derivatives, *as long as any substitution of values for the variables is deferred to the end.*

Example To evaluate the second partial derivatives of $f(x,y) = \ln|x+y^2|$ at $x = 1$, $y = 2$, we first compute the first partial derivatives of f:

$$f_x = \frac{1}{x+y^2}\frac{\partial}{\partial x}[x+y^2] = \frac{1}{x+y^2},$$

$$f_y = \frac{1}{x+y^2}\frac{\partial}{\partial y}[x+y^2] = \frac{2y}{x+y^2}.$$

Next, we differentiate each of these partial derivatives with respect to both x and y to obtain

$$\begin{aligned}
f_{xx} &= (f_x)_x \\
&= \left(\frac{1}{x+y^2}\right)_x \\
&= -\frac{1}{(x+y^2)^2}\frac{\partial}{\partial x}[x+y^2] \\
&= -\frac{1}{(x+y^2)^2}, \\
f_{xy} &= (f_x)_y \\
&= \left(\frac{1}{x+y^2}\right)_y \\
&= -\frac{1}{(x+y^2)^2}\frac{\partial}{\partial y}[x+y^2]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2y}{(x+y^2)^2}, \\
f_{yx} &= f_{xy} \\
&= -\frac{2y}{(x+y^2)^2}, \\
f_{yy} &= (f_y)_y \\
&= \left(\frac{2y}{x+y^2}\right)_y \\
&= \frac{(x+y^2)(2y)_y - 2y(x+y^2)_y}{(x+y^2)^2} \\
&= \frac{2(x+y^2) - 4y^2}{(x+y^2)^2} \\
&= \frac{2(x-y^2)}{(x+y^2)^2}.
\end{aligned}$$

Finally, we can evaluate these second partial derivatives at $x = 1$ and $y = 2$ to obtain

$$f_{xx}(1,2) = -\frac{1}{25}, \quad f_{xy}(1,2) = f_{yx}(1,2) = -\frac{4}{25}, \quad f_{yy}(1,2) = -\frac{6}{25}.$$

□

Example Let $f(x, y, z) = x^2y^4z^3$. We will compute the second partial derivatives of this function at the point $(x_0, y_0, z_0) = (-1, 2, 3)$ by repeated computation of first partial derivatives. First, we compute

$$f_x = (x^2y^4z^3)_x = (x^2)_x y^4z^3 = 2xy^4z^3,$$

by treating y and z as constants, then

$$f_y = (x^2y^4z^3)_y = (y^4)_y x^2z^3 = 4x^2y^3z^3,$$

by treating x and z as constants, and then

$$f_z = (x^2y^4z^3)_z = x^2y^4(z^3)_z = 3x^2y^4z^2.$$

We then differentiate each of these with respect to x, y and z to obtain the second partial derivatives:

$$\begin{aligned}
f_{xx} &= (f_x)_x = (2xy^4z^3)_x = 2y^4z^3, \\
f_{xy} &= (f_x)_y = (2xy^4z^3)_y = (2x)(4y^3)(z^3) = 8xy^3z^3,
\end{aligned}$$

$$\begin{aligned}
f_{xz} &= (f_x)_z = (2xy^4z^3)_z = (2xy^4)(3z^2) = 6xy^4z^2, \\
f_{yx} &= (f_y)_x = (4x^2y^3z^3)_x = 8xy^3z^3, \\
f_{yy} &= (f_y)_y = (4x^2y^3z^3)_y = (4x^2)(3y^2)(z^3) = 12x^2y^2z^3, \\
f_{yz} &= (f_y)_z = (4x^2y^3z^3)_z = (4x^2y^3)(3z^2) = 12x^2y^3z^2, \\
f_{zx} &= (f_z)_x = (3x^2y^4z^2)_x = 6xy^4z^2, \\
f_{zy} &= (f_z)_y = (3x^2y^4z^2)_y = (3x^2)(4y^3)(z^2) = 12x^2y^3z^2, \\
f_{zz} &= (f_z)_z = (3x^2y^4z^2)_z = (3x^2y^4)(2z) = 6x^2y^4z.
\end{aligned}$$

Then, these can be evaluated at (x_0, y_0, z_0) by substituting $x = -1$, $y = 2$, and $z = 3$ to obtain

$$f_{xx}(-1,2,3) = 864, \quad f_{xy}(-1,2,3) = -1728, \quad f_{xz}(-1,2,3) = -864,$$
$$f_{yx}(-1,2,3) = -1728, \quad f_{yy}(-1,2,3) = 1296, \quad f_{yz}(-1,2,3) = 864,$$
$$f_{zx}(-1,2,3) = -864, \quad f_{zy}(-1,2,3) = 864, \quad f_{zz}(-1,2,3) = 288.$$

Note that the order in which partial differentiation operations occur does not appear to matter; that is, $f_{xy} = f_{yx}$, for example. That is, Clairaut's Theorem applies for any number of variables. It also applies to any order of partial derivative. For example,

$$f_{xyy} = (f_{xy})_y = (8xy^3z^3)_y = 24xy^2z^3,$$
$$f_{yyx} = (f_{yy})_x = (12x^2y^2z^3)_x = 24xy^2z^3.$$

□

In single-variable calculus, *implicit differentiation* is applied to an equation that *implicitly* describes y as a function of x, in order to compute dy/dx. The same approach can be applied to an equation that implicitly describes any number of dependent variables in terms of any number of independent variables. The approach is the same as in the single-variable case: differentiate both sides of the equation with respect to the independent variable, leaving derivatives of dependent variables in the equation as unknowns. The resulting equation can then be solved for the unknown partial derivatives.

Example Consider the equation

$$x^2z + y^2z + z^2 = 1.$$

If we view this equation as one that implicitly describes z as a function of x and y, we can compute z_x and z_y using implicit differentiation with respect to x and y, respectively. Applying the Product Rule yields the equations

$$2xz + x^2z_x + y^2z_x + 2zz_x = 0,$$

$$x^2z_y + 2yz + y^2z_y + 2zz_y = 0,$$

which can then be solved for the partial derivatives to obtain

$$z_x = -\frac{2xz}{x^2 + y^2 + 2z}, \quad z_y = -\frac{2yz}{x^2 + y^2 + 2z}.$$

□

1.5 Tangent Planes, Linear Approximations and Differentiability

Now that we have learned how to compute partial derivatives of functions of several independent variables, in order to measure their instantaneous rates of change with respect to these variables, we will discuss another essential application of derivatives: the *approximation* of functions by *linear functions*. Linear functions are the simplest to work with, and for this reason, there are many instances in which functions are replaced by a linear approximation in the context of solving a problem such as solving a differential equation.

1.5.1 Tangent Planes and Linear Approximations

In single-variable calculus, we learned that the graph of a function $f(x)$ can be approximated near a point x_0 by its *tangent line*, which has the equation

$$y = f(x_0) + f'(x_0)(x - x_0).$$

For this reason, the function $L_f(x) = f(x_0) + f'(x_0)(x - x_0)$ is also referred to as the *linearization*, or *linear approximation*, of $f(x)$ at x_0.

Now, suppose that we have a function of two variables, $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, and a point $(x_0, y_0) \in D$. Furthermore, suppose that the first partial derivatives of f, f_x and f_y, exist at (x_0, y_0). Because the graph of this function is a *surface*, it follows that a linear function that approximates f near (x_0, y_0) would have a graph that is a *plane*.

Just as the tangent line of $f(x)$ at x_0 passes through the point $(x_0, f(x_0))$, and has a slope that is equal to $f'(x_0)$, the instantaneous rate of change of $f(x)$ with respect to x at x_0, a plane that best approximates $f(x, y)$ at (x_0, y_0) must pass through the point $(x_0, y_0, f(x_0, y_0))$, and the slope of the plane in the x- and y-directions, respectively, should be equal to the values of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

Since a general linear function of two variables can be described by the formula

$$L_f(x,y) = A(x-x_0) + B(y-y_0) + C,$$

so that $L_f(x_0,y_0) = C$, and a simple differentiation yields

$$\frac{\partial L_f}{\partial x} = A, \quad \frac{\partial L_f}{\partial y} = B,$$

we conclude that the linear function that best approximates $f(x,y)$ near (x_0,y_0) is the *linear approximation*

$$L_f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

Furthermore, the graph of this function is called the *tangent plane* of $f(x,y)$ at (x_0,y_0). Its equation is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

Example Let $f(x,y) = 2x^2y+3y^2$, and let $(x_0,y_0) = (1,1)$. Then $f(x_0,y_0) = 5$, and the first partial derivatives at (x_0,y_0) are

$$f_x(1,1) = 4xy|_{x=1,y=1} = 4, \quad f_y(1,1) = 2x^2 + 6y|_{x=1,y=1} = 8.$$

It follows that the tangent plane at $(1,1)$ has the equation

$$z - 5 = 4(x-1) + 8(y-1),$$

and the linearization of f at $(1,1)$ is

$$L_f(x,y) = 5 + 4(x-1) + 8(y-1).$$

Let $(x,y) = (1.1, 1.1)$. Then $f(x,y) = 6.292$, while $L_f(x,y) = 6.2$, for an error of $6.292 - 6.2 = 0.092$. However, if $(x,y) = (1.01, 1.01)$, then $f(x,y) = 5.120902$, while $L_f(x,y) = 5.12$, for an error of $5.120902 - 5.12 = 0.000902$. That is, moving 10 times as close to $(1,1)$ decreased the error by a factor of over 100. □

Another useful application of a linear approximation is to estimate the error in the value of a function, given estimates of error in its inputs. Given a function $z = f(x,y)$, and its linearization $L_f(x,y)$ around a point (x_0,y_0), if x_0 and y_0 are measured values and $dx = x - x_0$ and $dz = y - y_0$ are regarded as *errors* in x_0 and y_0, then the error in z can be estimated by computing

$$\begin{aligned} dz &= z - z_0 = L_f(x, y) - f(x_0, y_0) \\ &= [f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)] - f(x_0, y_0) \\ &= f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy. \end{aligned}$$

The variables dx and dy are called *differentials*, and dz is called the *total differential*, as it depends on the values of dx and dy. The total differential dz is only an *estimate* of the error in z; the *actual* error is given by $\Delta z = f(x, y) - f(x_0, y_0)$, when the actual errors in x and y, $\Delta x = x - x_0$ and $\Delta y = y - y_0$, are known. Since this is rarely the case in practice, one instead estimates the error in z from estimates dx and dy of the errors in x and y.

Example Recall that the volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Suppose that $r = 5$ cm and $h = 10$ cm. Then the volume is $V = 250\pi\,\text{cm}^3$. If the measurement error in r and h is at most 0.1 cm, then, to estimate the error in the computed volume, we first compute

$$V_r = 2\pi r h = 100\pi, \quad V_h = \pi r^2 = 25\pi.$$

It follows that the error in V is approximately

$$dV = V_r\, dr + V_h\, dh = 0.1(100\pi + 25\pi) = 12.5\pi\,\text{cm}^3.$$

If we specify $\Delta r = 0.1$ and $\Delta h = 0.1$, and compute the actual volume using radius $r + \Delta r = 5.1$ and height $h + \Delta h = 10.1$, we obtain

$$V + \Delta V = \pi(5.1)^2(10.1) = 262.701\pi\,\text{cm}^3,$$

which yields the actual error

$$\Delta V = 262.701\pi - 250\pi = 12.701\pi\,\text{cm}^3.$$

Therefore, the estimate of the error, dV, is quite accurate. □

1.5.2 Functions of More than Two Variables

The concepts of a tangent plane and linear approximation generalize to more than two variables in a straightforward manner. Specifically, given $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{p}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in D$, we define the *tangent* *space* of $f(x_1, x_2, \ldots, x_n)$ at $\mathbf{p}_0$ to be the n-dimensional *hyperplane* in $\mathbb{R}^{n+1}$ whose points $(x_1, x_2, \ldots, x_n, y)$ satisfy the equation

$$y - y_0 = \frac{\partial f}{\partial x_1}(\mathbf{p}_0)(x_1 - x_1^{(0)}) + \frac{\partial f}{\partial x_2}(\mathbf{p}_0)(x_2 - x_2^{(0)}) + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{p}_0)(x_n - x_n^{(0)}),$$

where $y_0 = f(\mathbf{p}_0)$. Similarly, the *linearization* of f at $\mathbf{p}_0$ is the function $L_f(x_1, x_2, \ldots, x_n)$ defined by

$$\begin{aligned} L_f(x_1, x_2, \ldots, x_n) &= y_0 + \frac{\partial f}{\partial x_1}(\mathbf{p}_0)(x_1 - x_1^{(0)}) + \frac{\partial f}{\partial x_2}(\mathbf{p}_0)(x_2 - x_2^{(0)}) + \\ &\quad \cdots + \frac{\partial f}{\partial x_n}(\mathbf{p}_0)(x_n - x_n^{(0)}). \end{aligned}$$

1.5.3 The Gradient Vector

It can be seen from the above definitions that writing formulas that involve the partial derivatives of functions of n variables can be cumbersome. This can be addressed by expressing collections of partial derivatives of functions of several variables using vectors and matrices, especially for vector-valued functions of several variables.

By convention, a point $\mathbf{p}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$, which can be identified with the *position vector* $\mathbf{p}_0 = \langle x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \rangle$, is considered to be a *column vector*

$$\mathbf{p}_0 = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}.$$

Also, by convention, given a function of n variables, $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, the collection of its partial derivatives with respect to all of its variables is written as a *row vector*

$$\nabla f(\mathbf{p}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{p}_0) & \frac{\partial f}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{p}_0) \end{bmatrix}.$$

This vector is called the *gradient* of f at $\mathbf{p}_0$.

Viewing the partial derivatives of f as a vector allows us to use vector operations to describe, much more concisely, the linearization of f. Specifically, the linearization of f at $\mathbf{p}_0$, evaluated at a point $\mathbf{p} = (x_1, x_2, \ldots, x_n)$, can be written as

$$L_f(\mathbf{p}) = f(\mathbf{p}_0) + \frac{\partial f}{\partial x_1}(\mathbf{p}_0)(x_1 - x_1^{(0)}) + \frac{\partial f}{\partial x_2}(\mathbf{p}_0)(x_2 - x_2^{(0)}) +$$

$$\begin{aligned} & \cdots + \frac{\partial f}{\partial x_n}(\mathbf{p}_0)(x_n - x_n^{(0)}) \\ = \quad & f(\mathbf{p}_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{p}_0)(x_i - x_i^{(0)}) \\ = \quad & f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0), \end{aligned}$$

where $\nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0)$ is the *dot product*, also known as the *inner product*, of the vectors $\nabla f(\mathbf{p}_0)$ and $\mathbf{p} - \mathbf{p}_0$. Recall that given two vectors $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$, the dot product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = 3x^2 y^3 z^4.$$

Then

$$\nabla f(x, y, z) = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} 6xy^3z^4 & 9x^2y^2z^4 & 12x^2y^3z^3 \end{bmatrix}.$$

Let $(x_0, y_0, z_0) = (1, 2, -1)$. Then

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \nabla f(1, 2, -1) \\ &= \begin{bmatrix} f_x(1, 2, -1) & f_y(1, 2, -1) & f_z(1, 2, -1) \end{bmatrix} \\ &= \begin{bmatrix} 48 & 36 & -96 \end{bmatrix}. \end{aligned}$$

It follows that the linearization of f at (x_0, y_0, z_0) is

$$\begin{aligned} L_f(x, y, z) &= f(1, 2, -1) + \nabla f(1, 2, -1) \cdot \langle x - 1, y - 2, z + 1 \rangle \\ &= 24 + \langle 48, 36, -96 \rangle \cdot \langle x - 1, y - 2, z + 1 \rangle \\ &= 24 + 48(x - 1) + 36(y - 2) - 96(z + 1) \\ &= 48x + 36y - 96z - 192. \end{aligned}$$

At the point $(1.1, 1.9, -1.1)$, we have $f(1.1, 1.9, -1.1) \approx 36.5$, while $L_f(1.1, 1.9, -1.1) = 34.8$. Because f is changing rapidly in all coordinate directions at $(1, 2, -1)$, it is not surprising that the linearization of f at this point is not highly accurate. □

1.5.4 The Jacobian Matrix

Now, let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a *vector-valued* function of n variables, with *component functions*

$$\mathbf{f}(\mathbf{p}) = \begin{bmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \\ \vdots \\ f_m(\mathbf{p}) \end{bmatrix},$$

where each $f_i : D \to \mathbb{R}^m$. Combining the two conventions described above, the partial derivatives of these component functions at a point $\mathbf{p}_0 \in D$ are arranged in an $m \times n$ *matrix*

$$J_{\mathbf{f}}(\mathbf{p}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}_0) & \frac{\partial f_1}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{p}_0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{p}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{p}_0) \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}_0) & \frac{\partial f_m}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{p}_0) \end{bmatrix}.$$

This matrix is called the *Jacobian matrix* of $\mathbf{f}$ at $\mathbf{p}_0$. It is also referred to as the *derivative* of $\mathbf{f}$ at $\mathbf{x}_0$, since it reduces to the scalar $f'(x_0)$ when f is a scalar-valued function of one variable. Note that rows of $J_{\mathbf{f}}(\mathbf{p}_0)$ correspond to component functions, and columns correspond to independent variables. This allows us to view $J_{\mathbf{f}}(\mathbf{p}_0)$ as the following collections of rows or columns:

$$J_{\mathbf{f}}(\mathbf{p}_0) = \begin{bmatrix} \nabla f_1(\mathbf{p}_0) \\ \nabla f_2(\mathbf{p}_0) \\ \vdots \\ \nabla f_m(\mathbf{p}_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{p}_0) & \frac{\partial \mathbf{f}}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{p}_0) \end{bmatrix}.$$

The Jacobian matrix provides a concise way of describing the linearization of a vector-valued function, just the gradient does for a scalar-valued function. The linearization of $\mathbf{f}$ at $\mathbf{p}_0$ is the function $\mathbf{L}_{\mathbf{f}}(\mathbf{p})$, defined by

$$\begin{aligned} \mathbf{L}_{\mathbf{f}}(\mathbf{p}) &= \begin{bmatrix} f_1(\mathbf{p}_0) \\ f_2(\mathbf{p}_0) \\ \vdots \\ f_m(\mathbf{p}_0) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{p}_0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{p}_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{p}_0) \end{bmatrix} (x_1 - x_1^{(0)}) + \cdots \\ &+ \begin{bmatrix} \frac{\partial f_1}{\partial x_n}(\mathbf{p}_0) \\ \frac{\partial f_2}{\partial x_n}(\mathbf{p}_0) \\ \vdots \\ \frac{\partial f_m}{\partial x_n}(\mathbf{p}_0) \end{bmatrix} (x_n - x_n^{(0)}) \end{aligned}$$

$$\begin{aligned} &= \mathbf{f}(\mathbf{p}_0) + \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{p}_0)(x_j - x_j^{(0)}) \\ &= \mathbf{f}(\mathbf{p}_0) + J_{\mathbf{f}}(\mathbf{p}_0)(\mathbf{p} - \mathbf{p}_0), \end{aligned}$$

where the expression $J_{\mathbf{f}}(\mathbf{p}_0)(\mathbf{p} - \mathbf{p}_0)$ involves *matrix multiplication* of the matrix $J_{\mathbf{f}}(\mathbf{p}_0)$ and the vector $\mathbf{p} - \mathbf{p}_0$. Note the similarity between this definition, and the definition of the linearization of a function of a single variable.

In general, given a $m \times n$ matrix A; that is, a matrix A with m rows and n columns, and an $n \times p$ matrix B, the product AB is the $m \times p$ matrix C, where the entry in row i and column j of C is obtained by computing the dot product of row i of A and column j of B. When computing the linearization of a vector-valued function $\mathbf{f}$ at the point $\mathbf{p}_0$ in its domain, the ith component function of the linearization is obtained by adding the value of the ith component function at $\mathbf{p}_0$, $f_i(\mathbf{p}_0)$, to the dot product of $\nabla f_i(\mathbf{p}_0)$ and the vector $\mathbf{p} - \mathbf{p}_0$, where $\mathbf{p}$ is the vector at which the linearization is to be evaluated.

Example Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{f}(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix} = \begin{bmatrix} e^x \cos y \\ e^{-2x} \sin y \end{bmatrix}.$$

Then the Jacobian matrix, or derivative, of $\mathbf{f}$ is the 2×2 matrix

$$J_{\mathbf{f}}(x, y) = \begin{bmatrix} \nabla f_1(x, y) \\ \nabla f_2(x, y) \end{bmatrix} = \begin{bmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ -2e^{-2x} \sin y & e^{-2x} \cos y \end{bmatrix}.$$

Let $(x_0, y_0) = (0, \pi/4)$. Then we have

$$J_{\mathbf{f}}(x_0, y_0) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

and the linearization of $\mathbf{f}$ at (x_0, y_0) is

$$\begin{aligned} \mathbf{L}_{\mathbf{f}}(x, y) &= \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix} + J_{\mathbf{f}}(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\sqrt{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x - 0 \\ y - \frac{\pi}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \\ \frac{\sqrt{2}}{2} - \sqrt{2}x + \frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) \end{bmatrix}. \end{aligned}$$

At the point $(x_1, y_1) = (0.1, 0.8)$, we have

$$\mathbf{f}(x_1, y_1) \approx \begin{bmatrix} 0.76998 \\ 0.58732 \end{bmatrix}, \quad \mathbf{L_f}(x_1, y_1) \approx \begin{bmatrix} 0.76749 \\ 0.57601 \end{bmatrix}.$$

Because of the relatively small partial derivatives at (x_0, y_0), the linearization at this point yields a fairly accurate approximation at (x_1, y_1). □

1.5.5 Differentiability

Before using a linearization to approximate a function near a point $\mathbf{p}_0$, it is helpful to know whether this linearization is actually an accurate approximation of the function in the first place. That is, we need to know if the function is *differentiable* at $\mathbf{p}_0$, which, informally, means that its instantaneous rate of change at $\mathbf{p}_0$ is well-defined. In the single-variable case, a function $f(x)$ is differentiable at x_0 if $f'(x_0)$ exists; that is, if the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In other words, we must have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0.$$

But $f(x_0) + f'(x_0)(x - x_0)$ is just the linearization of f at x_0, so we can say that f is differentiable at x_0 if and only if

$$\lim_{x \to x_0} \frac{f(x) - L_f(x)}{x - x_0} = 0.$$

Note that this is a *stronger* statement than simply requiring that

$$\lim_{x \to x_0} f(x) - L_f(x) = 0,$$

because as x approaches x_0, $|1/(x - x_0)|$ approaches ∞, so the difference $f(x) - L_f(x)$ must approach zero particularly rapidly in order for the fraction $[f(x) - L_f(x)]/(x - x_0)$ to approach zero. That is, the linearization must be a sufficiently accurate approximation of f near x_0 for this to be the case, in order for f to be differentiable at x_0.

This notion of differentiability is readily generalized to functions of several variables. Given $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and $\mathbf{p}_0 \in D$, we say that $\mathbf{f}$ is *differentiable* at $\mathbf{p}_0$ if

$$\lim_{\mathbf{p} \to \mathbf{p}_0} \frac{\|\mathbf{f}(\mathbf{p}) - \mathbf{L_f}(\mathbf{p})\|}{\|\mathbf{p} - \mathbf{p}_0\|} = 0,$$

where $\mathbf{L_f}(\mathbf{p})$ is the linearization of $\mathbf{f}$ at $\mathbf{p}_0$.

Example Let $f(x,y) = x^2y$. To verify that this function is differentiable at $(x_0, y_0) = (1,1)$, we first compute $f_x = 2xy$ and $f_y = x^2$. It follows that the linearization of f at $(1,1)$ is

$$\begin{aligned} L_f(x,y) &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= 1 + 2(x-1) + (y-1) = 2x + y - 2. \end{aligned}$$

Therefore, f is differentiable at $(1,1)$ if

$$\lim_{(x,y)\to(1,1)} \frac{|x^2y - (2x+y-2)|}{\|(x,y)-(1,1)\|} = \lim_{(x,y)\to(1,1)} \frac{|x^2y - (2x+y-2)|}{\sqrt{(x-1)^2+(y-1)^2}} = 0.$$

By rewriting this expression as

$$\frac{|x^2y - (2x+y-2)|}{\sqrt{(x-1)^2+(y-1)^2}} = \frac{|x-1||y(x+1)-2|}{\sqrt{(x-1)^2+(y-1)^2}},$$

and noting that

$$\lim_{(x,y)\to(1,1)} |y(x+1)-2| = 0, \quad 0 \leq \frac{|x-1|}{\sqrt{(x-1)^2+(y-1)^2}} \leq 1,$$

we conclude that the limit actually is zero, and therefore f is differentiable. □

There are three important conclusions that we can make regarding differentiable functions:

- If all partial derivatives of $\mathbf{f}$ at $\mathbf{p}_0$ exist, *and are continuous*, then $\mathbf{f}$ is differentiable at $\mathbf{p}_0$.
- Furthermore, if $\mathbf{f}$ is differentiable at $\mathbf{p}_0$, then it is continuous at $\mathbf{p}_0$. Note that the converse is not true; for example, $f(x) = |x|$ is continuous at $x = 0$, but it is not differentiable there, because $f'(x)$ does not exist there.
- If $\mathbf{f}$ is differentiable at $\mathbf{p}_0$, then its first partial derivatives exist at $\mathbf{p}_0$. This statement might seem redundant, because the first partial derivatives are used in the definition of the linearization, but it is important nonetheless, because the *converse* of this statement is not true. That is, if a function's first partial derivatives exist at a point, it is not necessarily differentiable at that point.

The notion of differentiability is related to not only partial derivatives, which only describe how a function changes as one of its variables changes, but also the instantaneous rate of change of a function as its variables change *along any direction*. If a function is differentiable at a point, that means its rate of change along any direction is well-defined. We will explore this idea further later in this chapter.

1.6 The Chain Rule

Recall from single-variable calculus that if a function $g(x)$ is differentiable at x_0, and $f(x)$ is differentiable at $g(x_0)$, then the derivative of the *composition* $(f \circ g)(x) = f(g(x))$ is given by the *Chain Rule*

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

We now generalize the Chain Rule to functions of several variables. Let $\mathbf{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and let $\mathbf{g} : U \subseteq \mathbb{R}^p \to D$. That is, the range of $\mathbf{g}$ is the domain of $\mathbf{f}$.

Assume that $\mathbf{g}$ is differentiable at a point $\mathbf{p}_0 \in U$, and that $\mathbf{f}$ is differentiable at the point $\mathbf{q}_0 = \mathbf{g}(\mathbf{p}_0)$. Then, $\mathbf{f}$ has a Jacobian matrix $J_{\mathbf{f}}(\mathbf{q}_0)$, and $\mathbf{g}$ has a Jacobian matrix $J_{\mathbf{g}}(\mathbf{p}_0)$. These matrices contain the first partial derivatives of $\mathbf{f}$ and $\mathbf{g}$ evaluated at $\mathbf{q}_0$ and $\mathbf{p}_0$, respectively.

Then, the Chain Rule states that the derivative of the composition $(\mathbf{f} \circ \mathbf{g}) : U \to \mathbb{R}^m$, defined by $(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))$, at $\mathbf{p}_0$, is given by the Jacobian matrix

$$J_{\mathbf{f} \circ \mathbf{g}}(\mathbf{p}_0) = J_{\mathbf{f}}(\mathbf{g}(\mathbf{p}_0))J_{\mathbf{g}}(\mathbf{p}_0).$$

That is, the derivative of $\mathbf{f} \circ \mathbf{g}$ at $\mathbf{p}_0$ is the product, in the sense of *matrix multiplication*, of the derivative of $\mathbf{f}$ at $\mathbf{g}(\mathbf{p}_0)$ and the derivative of $\mathbf{g}$ at $\mathbf{p}_0$. This is entirely analogous to the Chain Rule from single-variable calculus, in which the derivative of $f \circ g$ at x_0 is the product of the derivative of f at $g(x_0)$ and the derivative of g at x_0.

It follows from the rules of matrix multiplication that the partial derivative of the ith component function of $\mathbf{f} \circ \mathbf{g}$ with respect to the variable x_j, an independent variable of $\mathbf{g}$, is given by the *dot product* of the *gradient* of the ith component function of $\mathbf{f}$ with the vector that contains the partial derivatives of the component functions of $\mathbf{g}$ with respect to x_j. We now illustrate the application of this general Chain Rule with some examples.

Example Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = e^z \cos 2x \sin 3y,$$

and let $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ be a vector-valued function of one variable defined by

$$\mathbf{g}(t) = \langle x(t), y(t), z(t)\rangle = \langle 2t, t^2, t^3\rangle.$$

Then, $f \circ \mathbf{g}$ is a scalar-valued function of t,

$$(f \circ \mathbf{g})(t) = e^{z(t)} \cos 2x(t) \sin 3y(t) = e^{t^3} \cos 4t \sin 3t^2.$$

To compute its derivative with respect to t, we first compute

$$\nabla f = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} = \begin{bmatrix} -2e^z \sin 2x \sin 3y & 3e^z \cos 2x \cos 3y & e^z \cos 2x \sin 3y \end{bmatrix},$$

and

$$\mathbf{g}'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle 2, 2t, 3t^2\rangle,$$

and then apply the Chain Rule to obtain

$$\begin{aligned}
\frac{df}{dt} &= \nabla f(x(t), y(t), z(t)) \cdot \mathbf{g}'(t) \\
&= \begin{bmatrix} f_x(x(t), y(t), z(t)) & f_y(x(t), y(t), z(t)) & f_z(x(t), y(t), z(t)) \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} \\
&= f_x(x(t), y(t), z(t))\frac{dx}{dt} + f_y(x(t), y(t), z(t))\frac{dy}{dt} + f_z(x(t), y(t), z(t))\frac{dz}{dt} \\
&= (-2e^{z(t)} \sin 2x(t) \sin 3y(t))(2) + (3e^{z(t)} \cos 2x(t) \cos 3y(t))(2t) + \\
&\quad (e^{z(t)} \cos 2x(t) \sin 3y(t))(3t^2) \\
&= -4e^{t^3} \sin 4t \sin 3t^2 + 6te^{t^3} \cos 4t \cos 3t^2 + 3t^2 e^{t^3} \cos 4t \sin 3t^2.
\end{aligned}$$

□

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^2 y + xy^2,$$

and let $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{g}(s, t) = \begin{bmatrix} x(s, t) \\ y(s, t) \end{bmatrix} = \begin{bmatrix} 2s + t \\ s - 2t \end{bmatrix}.$$

Then, $f \circ \mathbf{g}$ is a scalar-valued function of s and t,

$$(f \circ \mathbf{g})(s, t) = x(s, t)^2 y(s, t) + x(s, t) y(s, t)^2 = (2s+t)^2(s-2t) + (2s+t)(s-2t)^2.$$

To compute its gradient, which includes its partial derivatives with respect to s and t, we first compute

$$\nabla f = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 2xy + y^2 & x^2 + 2xy \end{bmatrix},$$

and

$$J_{\mathbf{g}}(s,t) = \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix},$$

and then apply the Chain Rule to obtain

$$\begin{aligned}
\nabla(f \circ \mathbf{g})(s,t) &= \nabla f(x(s,t), y(s,t)) J_{\mathbf{g}}(s,t) \\
&= \begin{bmatrix} f_x(x(s,t), y(s,t)) & f_y(x(s,t), y(s,t)) \end{bmatrix} \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} \\
&= \left[\begin{array}{l} f_x(x(s,t), y(s,t))x_s + f_y(x(s,t), y(s,t))y_s \\ f_x(x(s,t), y(s,t))x_t + f_y(x(s,t), y(s,t))y_t \end{array} \right] \\
&= \left[\begin{array}{l} [2x(t)y(t) + y(t)^2](2) + [x(t)^2 + 2x(t)y(t)](1) \\ [2x(t)y(t) + y(t)^2](1) + [x(t)^2 + 2x(t)y(t)](-2) \end{array} \right] \\
&= \left[\begin{array}{l} 4(2s+t)(s-2t) + 2(s-2t)^2 + (2s+t)^2 + 2(2s+t)(s-2t) \\ 2(2s+t)(s-2t) + (s-2t)^2 - 2(2s+t)^2 - 4(2s+t)(s-2t) \end{array} \right].
\end{aligned}$$

□

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 + 2x^2,$$

and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(u,v) = \sin u \cos v.$$

Then $f \circ g$ is a scalar-valued function of u and v,

$$(f \circ g)(u,v) = (\sin u \cos v)^3 + 2(\sin u \cos v)^2.$$

To compute its gradient, which includes partial derivatives with respect to u and v, we first compute

$$f'(x) = 3x^2 + 4x,$$

and

$$\nabla g = \begin{bmatrix} g_u & g_v \end{bmatrix} = \begin{bmatrix} \cos u \cos v & -\sin u \sin v \end{bmatrix},$$

and then use the Chain Rule to obtain

$$\begin{aligned}
\nabla(f \circ \mathbf{g})(u,v) &= f'(g(u,v))\nabla g(u,v) \\
&= [3(g(u,v))^2 + 4g(u,v)] \begin{bmatrix} \cos u \cos v & -\sin u \sin v \end{bmatrix} \\
&= [3\sin^2 u \cos^2 v + 4 \sin u \cos v] \begin{bmatrix} \cos u \cos v & -\sin u \sin v \end{bmatrix} \\
&= \begin{bmatrix} (3\sin^2 u \cos^2 v + 4\sin u \cos v)\cos u \cos v & -(3\sin^2 u \cos^2 v + 4 \sin u \cos v)\sin u \sin v \end{bmatrix}.
\end{aligned}$$

□

Example Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 y \\ xy^2 \end{bmatrix},$$

and let $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\mathbf{g}(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle.$$

Then $\mathbf{f} \circ \mathbf{g}$ is a vector-valued function of t,

$$\mathbf{f}(t) = \langle \cos^2 t \sin t, \cos t \sin^2 t \rangle.$$

To compute its derivative with respect to t, we first compute

$$J_{\mathbf{f}}(x,y) = \begin{bmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ y^2 & 2xy \end{bmatrix},$$

and $\mathbf{g}'(t) = \langle -\sin t, \cos t \rangle$, and then use the Chain Rule to obtain

$$\begin{aligned}
(\mathbf{f} \circ \mathbf{g})'(t) &= J_{\mathbf{f}}(x(t), y(t))\mathbf{g}'(t) = \begin{bmatrix} (f_1)_x(x(t),y(t)) & (f_1)_y(x(t),y(t)) \\ (f_2)_x(x(t),y(t)) & (f_2)_y(x(t),y(t)) \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \\
&= \big\langle (f_1)_x(x(t),y(t))x'(t) + (f_1)_y(x(t),y(t))y'(t), (f_2)_x(x(t),y(t))x'(t) + (f_2)_y(x(t),y(t))y'(t) \big\rangle \\
&= \langle 2\cos t \sin t(-\sin t) + \cos^2 t(\cos t), \sin^2 t(-\sin t) + 2\cos t \sin t(\cos t) \rangle \\
&= \langle -2\cos t \sin^2 t + \cos^3 t, -\sin^3 t + 2\cos^2 t \sin t \rangle.
\end{aligned}$$

□

1.6.1 The Implicit Function Theorem

The Chain Rule can also be used to compute partial derivatives of implicitly defined functions in a more convenient way than is provided by implicit differentiation. Let the equation $F(x,y) = 0$ implicitly define y as a differentiable function of x. That is, $y = f(x)$ where $F(x, f(x)) = 0$ for x in the domain of f. If F is differentiable, then, by the Chain Rule, we can differentiate the equation $F(x, y(x)) = 0$ with respect to x and obtain

$$F_x + F_y \frac{dy}{dx} = 0,$$

which yields

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

By the *Implicit Function Theorem*, the equation $F(x, y) = 0$ defines y implicitly as a function of x near (x_0, y_0), where $F(x_0, y_0) = 0$, provided that $F_y(x_0, y_0) \neq 0$ and F_x and F_y are continuous near (x_0, y_0). Under these conditions, we see that dy/dx is defined at (x_0, y_0) as well.

Example Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^2 + y^2 - 4.$$

The equation $F(x, y) = 0$ defines y implicitly as a function of x, provided that F satisfies the conditions of the Implicit Function Theorem.

We have

$$F_x = 2x, \quad F_y = 2y.$$

Since both of these partial derivatives are polynomials, and therefore are continuous on all of $\mathbb{R}^2$, it follows that if $F_y \neq 0$, then y can be implicitly defined as a function of x at a point where $F(x, y, z) = 0$, and

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{x}{y}.$$

For example, at the point $(x, y) = (0, 2)$, $F(x, y) = 0$, and $F_y = 4$. Therefore, y can be implicitly defined as a function of x near this point, and at $x = 0$, we have $dy/dx = 0$. □

More generally, let $F : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$, and let $\mathbf{p}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}, y^{(0)}) \in D$ be such that $F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}, y^{(0)}) = 0$. In this case, the Implicit Function Theorem states that if $F_y \neq 0$ near $\mathbf{p}_0$, and all first partial derivatives of F are continuous near $\mathbf{p}_0$, then this equation defines y as a function of $x_1, x_2, \ldots, x_n$, and

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}, \quad i = 1, 2, \ldots, n.$$

To see this, we differentiate the equation $F(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}, y^{(0)}) = 0$ with respect to x_i to obtain the equation

$$F_{x_i} + F_y \frac{\partial y}{\partial x_i} = 0,$$

where all partial derivatives are evaluated at $\mathbf{p}_0$, and solve for $\partial y/\partial x_i$ at $\mathbf{p}_0$.

Example Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$F(x, y, z) = x^2 z + z^2 y - 2xyz + 1.$$

The equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y, provided that F satisfies the conditions of the Implicit Function Theorem.

We have

$$F_x = 2xz - 2yz, \quad F_y = z^2 - 2xz, \quad F_z = x^2 + 2yz - 2xy.$$

Since all of these partial derivatives are polynomials, and therefore are continuous on all of $\mathbb{R}^3$, it follows that if $F_z \neq 0$, then z can be implicitly defined as a function of x and y at a point where $F(x, y, z) = 0$, and

$$z_x = -\frac{F_x}{F_z} = \frac{2yz - 2xz}{x^2 + 2yz - 2xy}, \quad z_y = -\frac{F_y}{F_z} = \frac{2xz - z^2}{x^2 + 2yz - 2xy}.$$

For example, at the point $(x, y, z) = (1, 0, -1)$, $F(x, y, z) = 0$, and $F_z = 1$. Therefore, z can be implicitly defined as a function of x and y near this point, and at $(x, y) = (1, 0)$, we have $z_x = 2$ and $z_y = -3$. □

We now consider the most general case: let $\mathbf{F} : D \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^m$, and let

$$\mathbf{p}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}, y_1^{(0)}, y_2^{(0)}, \ldots, y_m^{(0)}) \in D$$

be such that

$$\mathbf{F}(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}, y_1^{(0)}, y_2^{(0)}, \ldots, y_m^{(0)}) = \mathbf{0}.$$

If we differentiate this *system* of equations with respect to x_i, we obtain the *systems* of *linear* equations

$$\mathbf{F}_{x_i} + \mathbf{F}_{y_1} \frac{\partial y_1}{\partial x_i} + \mathbf{F}_{y_2} \frac{\partial y_2}{\partial x_i} + \cdots + \mathbf{F}_{y_m} \frac{\partial y_m}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n,$$

where all partial derivatives are evaluated at $\mathbf{p}_0$.

To examine the solvability of these systems of equations, we first define $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$, and denote the component functions of the vector-valued function $\mathbf{F}$ by $\mathbf{F} = \langle F_1, F_2, \ldots, F_m \rangle$. We then define the Jacobian matrices

$$J_{\mathbf{x},\mathbf{F}}(\mathbf{p}_0) = \left[\begin{array}{cccc} \frac{\partial F_1}{\partial x_1}(\mathbf{p}_0) & \frac{\partial F_1}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{p}_0) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{p}_0) & \frac{\partial F_2}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{p}_0) \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial F_m}{\partial x_1}(\mathbf{p}_0) & \frac{\partial F_m}{\partial x_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_m}{\partial x_n}(\mathbf{p}_0) \end{array} \right],$$

$$J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0) = \left[\begin{array}{cccc} \frac{\partial F_1}{\partial y_1}(\mathbf{p}_0) & \frac{\partial F_1}{\partial y_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_1}{\partial y_m}(\mathbf{p}_0) \\ \frac{\partial F_2}{\partial y_1}(\mathbf{p}_0) & \frac{\partial F_2}{\partial y_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_2}{\partial y_m}(\mathbf{p}_0) \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\mathbf{p}_0) & \frac{\partial F_m}{\partial y_2}(\mathbf{p}_0) & \cdots & \frac{\partial F_m}{\partial y_m}(\mathbf{p}_0) \end{array} \right],$$

and

$$J_{\mathbf{y}}(\mathbf{x}_0) = \left[\begin{array}{cccc} \frac{\partial y_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial y_1}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial y_1}{\partial x_n}(\mathbf{x}_0) \\ \frac{\partial y_2}{\partial x_1}(\mathbf{x}_0) & \frac{\partial y_2}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial y_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \cdots & \cdots & \vdots \\ \frac{\partial y_m}{\partial x_1}(\mathbf{x}_0) & \frac{\partial y_m}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial y_1}{\partial x_n}(\mathbf{x}_0) \end{array} \right].$$

Then, from our previous differentiation with respect to x_i, for each $i = 1, 2, \ldots, n$, we can concisely express our systems of equations as a single system

$$J_{\mathbf{x},\mathbf{F}}(\mathbf{p}_0) + J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0) J_{\mathbf{y}}(\mathbf{x}_0) = \mathbf{0}.$$

If the matrix $J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0)$ is *invertible* (also *nonsingular*), which is the case if and only if its determinant is nonzero, and if all first partial derivatives of $\mathbf{F}$ are continuous near $\mathbf{p}_0$, then the equation $\mathbf{F}(\mathbf{p}) = \mathbf{0}$ implicitly defines $y_1, y_2, \ldots, y_m$ as a function of $x_1, x_2, \ldots, x_n$, and

$$J_{\mathbf{y}}(\mathbf{x}_0) = -[J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0)]^{-1} J_{\mathbf{x},\mathbf{F}}(\mathbf{p}_0),$$

where $[J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0)]^{-1}$ is the *inverse* of the matrix $J_{\mathbf{y},\mathbf{F}}(\mathbf{p}_0)$.

Example Let $\mathbf{F} : \mathbb{R}^4 \to \mathbb{R}^2$ by defined by

$$\mathbf{F}(x, y, u, v) = \left[\begin{array}{c} F_1(x, y, u, v) \\ F_2(x, y, u, v) \end{array} \right] = \left[\begin{array}{c} xu + y^2 v \\ x^2 v + yu + 1 \end{array} \right].$$

Then the vector equation $\mathbf{F}(x, y, u, v) = \mathbf{0}$ implicitly defines (u, v) as a function of (x, y), provided that $\mathbf{F}$ satisifes the conditions of the Implicit

Function Theorem. We will compute the partial derivatives of u and v with respect to x and y, at a point that satisfies this equation.

We have

$$J_{(x,y),\mathbf{F}}(x,y,u,v) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} u & 2yv \\ 2xv & u \end{bmatrix},$$

$$J_{(u,v),\mathbf{F}}(x,y,u,v) = \begin{bmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{bmatrix} = \begin{bmatrix} x & y^2 \\ y & x^2 \end{bmatrix}.$$

From the formula for the inverse of a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we obtain

$$\begin{aligned} J_{(u,v)}(x,y) &= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \\ &= -[J_{(u,v),\mathbf{F}}(x,y,u,v)]^{-1} J_{(x,y),\mathbf{F}}(x,y,u,v) \\ &= -\frac{1}{x^3-y^3} \begin{bmatrix} x^2 & -y^2 \\ -y & x \end{bmatrix} \begin{bmatrix} u & 2yv \\ 2xv & u \end{bmatrix} \\ &= \frac{1}{y^3-x^3} \begin{bmatrix} x^2u - 2xy^2v & 2x^2yv - y^2u \\ 2x^2v - yu & xu - 2y^2v \end{bmatrix}. \end{aligned}$$

These partial derivatives can then be evaluated at any point (x,y,u,v) such that $\mathbf{F}(x,y,u,v) = \mathbf{0}$, such as $(x,y,u,v) = (0,1,0,-1)$. Note that the matrix $J_{(u,v),\mathbf{F}}(x,y,u,v)$ is not invertible (that is, *singular*) if its determinant $x^3 - y^3 = 0$; that is, if $x = y$. When this is the case, (u,v) can not be implicitly defined as a function of (x,y). □

1.7 Directional Derivatives and the Gradient Vector

Previously, we defined the gradient as the vector of all of the first partial derivatives of a scalar-valued function of several variables. Now, we will learn about how to use the gradient to measure the rate of change of the function with respect to a change of its variables in *any* direction, as opposed to a change in a single variable. This is extremely useful in applications in which the minimum or maxmium value of a function is sought. We will also learn how the gradient can be used to easily describe tangent planes to level surfaces, thus providing an alternative to implicit differentiation or the Chain Rule.

1.7.1 The Gradient Vector

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of n variables $x_1, x_2, \ldots, x_n$. Recall that the vector of its first partial derivatives,

$$\nabla f = \begin{bmatrix} f_{x_1} & f_{x_2} & \cdots & f_{x_n} \end{bmatrix},$$

is called the *gradient* of f.

Example Let $f(x, y, z) = e^{-(x^2+y^2)} \cos z$. Then

$$\nabla f = \begin{bmatrix} -2xe^{-(x^2+y^2)} \cos z & -2ye^{-(x^2+y^2)} \cos z & -e^{-(x^2+y^2)} \sin z \end{bmatrix}.$$

Therefore, at the point $(x_0, y_0, z_0) = (1, 2, \pi/3)$, the gradient is the vector

$$\begin{aligned} \nabla f(x_0, y_0, z_0) &= \begin{bmatrix} f_x(1, 2, \pi/3) & f_y(1, 2, \pi/3) & f_z(1, 2, \pi/3) \end{bmatrix} \\ &= \left\langle -e^{-5}, -2e^{-5}, -\frac{\sqrt{3}}{2} e^{-5} \right\rangle. \end{aligned}$$

□

It should be noted that various differentiation rules from single-variable calculus have direct generalizations to the gradient. Let u and v be differentiable functions defined on $\mathbb{R}^n$. Then, we have:

- *Linearity:*

$$\nabla(au + bv) = a\nabla u + b\nabla v$$

  where a and b are constants

- *Product Rule:*

$$\nabla(uv) = u\nabla v + v\nabla u$$

- *Quotient Rule:*

$$\nabla \left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$$

- *Power Rule:*

$$\nabla u^n = nu^{n-1}\nabla u$$

1.7.2 Directional Derivatives

The components of the gradient vector ∇f represent the instantaneous rates of change of the function f with respect to any *one* of its independent variables. However, in many applications, it is useful to know how f changes as its variables change along *any* path from a given point. To that end, given $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, and a *unit* vector $\mathbf{u} = \langle a, b\rangle \in \mathbb{R}^2$, we define the *directional derivative* of f at $(x_0, y_0) \in D$ in the direction of $\mathbf{u}$ to be

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}.$$

When $\mathbf{u} = \mathbf{i} = \langle 1, 0\rangle$, then $D_{\mathbf{u}}f = f_x$, and when $\mathbf{u} = \mathbf{j} = \langle 0, 1\rangle$, then $D_{\mathbf{u}}f = f_y$. For general $\mathbf{u}$, $D_{\mathbf{u}}f(x_0, y_0)$ represents the instantaneous rate of change of f as (x, y) change in the direction of $\mathbf{u}$ from the point (x_0, y_0).

Because it is cumbersome to compute a directional derivative using the definition directly, it is desirable to be able to relate the directional derivative to the partial derivatives, which can be computed easily using differentiation rules. We have

$$\begin{aligned}
D_{\mathbf{u}}f(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0 + bh) + f(x_0, y_0 + bh) - f(x_0, y_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0 + bh)}{h} + \\
&\quad \frac{f(x_0, y_0 + bh) - f(x_0, y_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0 + bh)}{ah} a + \\
&\quad \frac{f(x_0, y_0 + bh) - f(x_0, y_0)}{bh} b \\
&= f_x(x_0, y_0)a + f_y(x_0, y_0)b \\
&= \nabla f(x_0, y_0) \cdot \mathbf{u}.
\end{aligned}$$

That is, the directional derivative in the direction of $\mathbf{u}$ is the *dot product* of the gradient with $\mathbf{u}$. It can be shown that this is the case for any number of variables: given $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, and a unit vector $\mathbf{u} \in \mathbb{R}^n$, the directional derivative of f at $\mathbf{x}_0 \in \mathbb{R}^n$ in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u}.$$

Because the dot product $\mathbf{a} \cdot \mathbf{b}$ can also be defined as

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, the directional derivative can be used to determine the direction along which f increases most rapidly, decreases most rapidly, or does not change at all.

We first note that if θ is the angle between $\nabla f(\mathbf{x}_0)$ and $\mathbf{u}$, then

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u} = \|\nabla f(\mathbf{x}_0)\| \cos\theta.$$

Then we have the following:

- When $\theta = 0$, $\cos\theta = 1$, so $D_{\mathbf{u}} f$ is maximized, and its value is $\|\nabla f(\mathbf{x}_0)\|$. In this case,

  $$\mathbf{u} = \frac{\nabla f(\mathbf{x}_0)}{\|\nabla f(\mathbf{x}_0)\|},$$

  and this is called the *direction of steepest ascent.*

- When $\theta = \pi$, $\cos\theta = -1$, so $D_{\mathbf{u}} f$ is minimized, and its value is $-\|\nabla f(\mathbf{x}_0)\|$. In this case,

  $$\mathbf{u} = -\frac{\nabla f(\mathbf{x}_0)}{\|\nabla f(\mathbf{x}_0)\|},$$

  and this is called the *direction of steepest descent.*

- When $\theta = \pm\pi/2$, $\cos\theta = 0$, so $D_{\mathbf{u}} = 0$. In this case, $\mathbf{u}$ is a unit vector that is *orthogonal* (perpendicular) to $\nabla f(\mathbf{x}_0)$. Since f is not changing at all along this direction, it follows that $\mathbf{u}$ indicates the direction of a *level set* of f, on which $f(\mathbf{x}) = f(\mathbf{x}_0)$.

The direction of steepest descent is of particular interest in applications in which the goal is to find the minimum value of f. From a starting point $\mathbf{x}_0$, one can choose a new point $\mathbf{x}_1 = \mathbf{x}_0 + \alpha\mathbf{u}$, where $\mathbf{u} = -\nabla f(\mathbf{x}_0)$ is the direction of steepest descent, by choosing α so as to minimize $f(\mathbf{x}_1)$. Then, this process can be repeated using the direction of steepest descent at $\mathbf{x}_1$, which is $-\nabla f(\mathbf{x}_1)$, to compute a new point $\mathbf{x}_2$, and so on, until a minimum is found. This process is called the *method of steepest descent.*

While not used very often in practice, it serves as a useful building block for some of the most powerful methods that are used in practice for minimizing functions.

Example Let $f(x, y) = x^2y + y^3$, and let $(x_0, y_0) = (2, -2)$. Then

$$\nabla f(x, y) = \left[\begin{array}{cc} f_x(x, y) & f_y(x, y) \end{array}\right] = \left[\begin{array}{cc} 2xy & x^2 + 3y^2 \end{array}\right],$$

which yields $\nabla f(x_0, y_0) = \langle f_x(2, -2), f_y(2, -2)\rangle = \langle -8, 16\rangle$. It follows that the direction of steepest ascent is

$$\mathbf{u} = \frac{\nabla f(2, -2)}{\|\nabla f(2, -2)\|} = \frac{\langle -8, 16\rangle}{\sqrt{(-8)^2 + 16^2}} = \frac{\langle -8, 16\rangle}{\sqrt{320}} = \frac{\langle -8, 16\rangle}{8\sqrt{5}} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle.$$

For this $\mathbf{u}$, we have $D_{\mathbf{u}}f(2, -2) = \|\nabla f(2, -2)\| = 8\sqrt{5}$.

Furthermore, the direction of steepest descent is

$$\mathbf{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle,$$

and along this direction, we have $D_{\mathbf{u}}f(2, -2) = -\|\nabla f(2, -2)\| = -8\sqrt{5}$. Finally, the directions along which f does not change at all are those that are orthogonal to the directions of steepest ascent and descent,

$$\mathbf{u} = \pm\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle.$$

The level curve defined by the equation $f(x, y) = f(2, -2) = -16$ proceeds along these directions from the point $(2, -2)$. □

1.7.3 Tangent Planes to Level Surfaces

Let $F : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a function of three variables x, y and z that implicitly defines a surface through the equation $F(x, y, z) = 0$, and let (x_0, y_0, z_0) be a point on that surface. If F satisfies the conditions of the Implicit Function Theorem at (x_0, y_0, z_0), then the equation of the plane that is tangent to the surface at this point can be obtained using the fact that z is implicitly defined as a function of x and y near this point. It then follows that the equation of the tangent plane is

$$z - z_0 = z_x(x_0, y_0)(x - x_0) + z_y(x_0, y_0)(y - y_0),$$

where, by the Chain Rule,

$$z_x(x_0, y_0) = -\frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}, \quad z_y(x_0, y_0) = -\frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}.$$

This is not possible if $F_z(x_0, y_0, z_0) = 0$, because then the Implicit Function Theorem does not apply.

It would be desirable to be able to obtain the equation of the tangent plane even if $F_z(x_0, y_0, z_0) = 0$, because the level surface still has a tangent plane at that point even if z cannot be implicitly defined as a function of x and y. To that end, we note that any direction $\mathbf{u}$ within the tangent plane is parallel to the tangent vector of some curve that lies within the surface and passes through (x_0, y_0, z_0). Because $F(x, y, z) = 0$ on this surface, it follows that $D_{\mathbf{u}}F(x_0, y_0, z_0) = 0$. However, this implies that $\nabla F(x_0, y_0, z_0)$ must be orthogonal to $\mathbf{u}$, in view of

$$D_{\mathbf{u}}F(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) \cdot \mathbf{u} = 0.$$

Since this is the case for *any* direction $\mathbf{u}$ within the tangent plane, we conclude that $\nabla F(x_0, y_0, z_0)$ is *normal* to the tangent plane, and therefore the equation of this plane is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Note that this equation is equivalent to that obtained using the Chain Rule, when $F_z(x_0, y_0, z_0) \neq 0$.

The gradient not only provides the normal vector to the tangent plane, but also the direction numbers of the *normal line* to the surface at (x_0, y_0, z_0), which is the line that passes through the surface at this point and is perpendicular to the tangent plane. The equation of this line, in parametric form, is

$$x = x_0 + tF_x(x_0, y_0, z_0), \quad y = y_0 + tF_y(x_0, y_0, z_0), \quad z = z_0 + tF_z(x_0, y_0, z_0).$$

Example Let $F(x, y, z) = x^2 + y^2 + z^2 - 2x - 4y - 4$. Then the equation $F(x, y, z) = 0$ defines a sphere of radius 3 centered at $(1, 2, 0)$. At the point $(x_0, y_0, z_0) = (3, 3, 2)$, we have

$$\begin{aligned} \nabla F(x_0, y_0, z_0) &= \begin{bmatrix} F_x(x_0, y_0, z_0) & F_y(x_0, y_0, z_0) & F_z(x_0, y_0, z_0) \end{bmatrix} \\ &= \begin{bmatrix} 2x_0 - 2 & 2y_0 - 4 & 2z_0 \end{bmatrix} \\ &= \langle 4, 2, 4 \rangle. \end{aligned}$$

It follows that the equation of the plane that is tangent to the sphere at $(3, 3, 2)$ is

$$4(x - x_0) + 2(y - y_0) + 4(z - z_0) = 0,$$

and the equation of the normal line, in parametric form, is

$$\begin{aligned} x &= x_0 + tF_x(x_0, y_0, z_0) = 3 + 4t, \\ y &= y_0 + tF_y(x_0, y_0, z_0) = 3 + 2t, \\ z &= z_0 + tF_z(x_0, y_0, z_0) = 2 + 4t. \end{aligned}$$

Equivalently, we can describe the normal line using its symmetric equations,

$$\frac{x-3}{4} = \frac{y-3}{2} = \frac{z-2}{4}.$$

□

When $F_z \neq 0$ at (x_0, y_0, z_0), two linearly independent vectors within a tangent plane can easily be obtained by first setting $x - x_0 = 1$ and $y - y_0 = 0$ in the equation of the tangent plane to obtain the first vector, and then $x - x_0 = 0$ and $y - y_0 = 1$ to obtain the second vector. For both vectors, the initial point is the point of tangency (x_0, y_0, z_0).

Recall the equation of the tangent plane, in its most general form,

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0,$$

where, for brevity, we assume that all partial derivatives are evaluated at (x_0, y_0, z_0). Setting $x - x_0 = 1$ and $y - y_0 = 0$ yields

$$F_x + F_z(z - z_0) = 0,$$

which can be rearranged to obtain

$$z - z_0 = -\frac{F_x}{F_z} = z_x.$$

It follows that the vector $\langle 1, 0, z_x \rangle$ is a vector that lies within the tangent plane.

Similarly, by setting $x - x_0 = 0$ and $y - y_0 = 1$, it follows that the vector $\langle 0, 1, z_y \rangle$ is also within the tangent plane. Because these vectors cannot be scalar multiples of each other, they must be linearly independent. It can be verified that both of these vectors are orthogonal to the normal to the tangent plane, $\nabla F(x_0, y_0, z_0)$. For example, we have

$$\langle 1, 0, z_x \rangle \cdot \langle F_x, F_y, F_z \rangle = F_x + F_z z_x = F_x + F_z(-F_x/F_z) = 0.$$

1.7.4 Tangent Lines to Level Curves

The preceding discussion about tangent planes to level surfaces can be scaled down to two dimensions, yielding equations of tangent lines to level curves. Consider a curve defined by the equation $F(x, y) = 0$. At a point (x_0, y_0) on this curve, the tangent vector is pointing in a direction $\mathbf{u}$ such that

$$D_{\mathbf{u}}F(x_0, y_0) = \nabla F(x_0, y_0) \cdot \mathbf{u} = 0,$$

since F is equal to the constant function 0 along the curve. That is, $\nabla F(x_0, y_0)$ is orthogonal to the tangent line to the curve at (x_0, y_0). Therefore, for any point (x, y) on the tangent line, we have

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0.$$

Thus we have an equation for the tangent line that is valid even when $dy/dx = -F_x/F_y$ is not defined due to $F_y(x_0, y_0) = 0$, which is analogous to the most general form of the equation of a tangent plane that is valid even when $F_z(x_0, y_0, z_0) = 0$.

When $F_y(x_0, y_0) \neq 0$, a vector contained within this tangent line can be obtained by setting $x - x_0 = 1$ in the above equation of the tangent plane, which yields the vector

$$\langle x - x_0, y - y_0 \rangle = \langle 1, -F_x/F_y \rangle = \langle 1, dy/dx \rangle$$

with initial point (x_0, y_0). It can then be verified that this vector is orthogonal to $\nabla F(x_0, y_0)$. We have

$$\nabla F \cdot \langle 1, dy/dx \rangle = \langle F_x, F_y \rangle \cdot \langle 1, -F_x/F_y \rangle = F_x + F_y(-F_x/F_y) = 0,$$

where it is assumed that all derivatives are evaluated at (x_0, y_0).

1.8 Maximum and Minimum Values

In single-variable calculus, one learns how to compute maximum and minimum values of a function. We first recall these methods, and then we will learn how to generalize them to functions of several variables.

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. A *local maximum* of a function f is a point $\mathbf{a} \in D$ such that $f(\mathbf{x}) \leq f(\mathbf{a})$ for $\mathbf{x}$ near $\mathbf{a}$. The value $f(\mathbf{a})$ is called a *local maximum value*. Similarly, f has a *local minimum* at $\mathbf{a}$ if $f(\mathbf{x}) \geq f(\mathbf{a})$ for $\mathbf{x}$ near $\mathbf{a}$, and the value $f(\mathbf{a})$ is called a *local minimum value*.

When a function of a single variable, $f(x)$, has a local maximum or minimum at $x = a$, then a must be a *critical point* of f, which means that $f'(c) = 0$, or f' does not exist at a (which is the case if, for example, the graph of f has a sharp corner at a). In general, if f is differentiable at a point $\mathbf{a}$, then in order for $\mathbf{a}$ to be a local maximum or minimum of f, the rate of change of f, as its independent variables change *in any direction* $\mathbf{u}$, must be zero. That is, we must have $\nabla f(\mathbf{a}) \cdot \mathbf{u} = 0$ for any unit vector $\mathbf{u}$. The only way to ensure this is to require that $\nabla f(\mathbf{a}) = \mathbf{0}$. Therefore, we say that $\mathbf{a}$ is a *critical point* if $\nabla f(\mathbf{a}) = \mathbf{0}$ or if any partial derivative of f does not exist at $\mathbf{a}$.

Once we have found the critical points of a function, we must determine whether they correspond to local maxima or minima. In the single-variable case, we can use the *Second Derivative Test*, which states that if a is a critical point of f, and $f''(a) > 0$, then a is a local minimum, while if $f''(a) < 0$, a is a local maximum, and if $f''(a) = 0$, the test is inconclusive.

This test is generalized to the multivariable case as follows: first, we form the *Hessian*, which is the *matrix* of second partial derivatives at $\mathbf{a}$. If f is a function of n variables, then the Hessian is an $n \times n$ matrix H, and the entry in row i, column j of H is defined by

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}).$$

Because mixed second partial derivatives are equal if they are continuous, it follows that H is a *symmetric* matrix, meaning that $H_{ij} = H_{ji}$.

We can now state the *Second Derivatives Test*. If $\mathbf{a}$ is a critical point of f, and the Hessian, H, is *positive definite* at $\mathbf{a}$, then $\mathbf{a}$ is a local minimum of f. The notion of a matrix being positive definite is the generalization to matrices of the notion of a positive number. When a matrix H is symmetric and positive definite, the following statements are all true:

- $\mathbf{x}^T H\mathbf{x} > 0$, where $\mathbf{x}$ is a nonzero column vector of real numbers, and $\mathbf{x}^T$ is the *transpose* of $\mathbf{x}$, which is a row vector. Note that $\mathbf{x}^T H\mathbf{x}$ is the same as $\mathbf{x} \cdot (H\mathbf{x})$.
- The eigenvalues of H are positive.
- The determinant of H is positive.
- The diagonal entries of H, H_{ii} for $i = 1, 2, \ldots, n$, are positive.

If a curve contained within the graph of f passes through a critical point $\mathbf{a}$ with unit tangent vector $\mathbf{u}$, then the concavity of the curve at $\mathbf{a}$ is measured by $\mathbf{u}^T H\mathbf{u}$). If H is positive definite, then this concavity is positive for any unit tangent vector $\mathbf{u}$, which means that f has a local minimum at $\mathbf{a}$.

On the other hand, if H is *negative definite*, then f has a local maximum at $\mathbf{a}$. This means that $\mathbf{x}^T H\mathbf{x} < 0$ for any nonzero real vector $\mathbf{x}$, and that the eigenvalues and diagonal entries of H are negative. However, the determinant is not necessarily negative. Because it is equal to the product of the eigenvalues, the determinant is *positive* of n is even, and negative if n is odd. When H is negative definite at $\mathbf{a}$, then any curve contained within the graph of f that passes through $\mathbf{a}$ is concave down at $\mathbf{a}$. That is, $\mathbf{u}^T H\mathbf{u} < 0$, where $\mathbf{u}$ is the unit tangent vector of the curve at $\mathbf{a}$.

If H is *indefinite*, which is the case if it is neither positive definite nor negative definite, and therefore has both positive and negative eigenvalues, then we say that f has a *saddle point* at $\mathbf{a}$. This means that the graph of f crosses its tangent plane at $\mathbf{a}$, and the term "saddle point" arises from the fact that curves contained within the graph of f that pass through $\mathbf{a}$ can be concave up along some directions and concave down along orthogonal directions, thus giving the graph the appearance of a saddle.

Finally, if H is a *singular* matrix, meaning that one of its eigenvalues, and therefore its determinant, is equal to zero, the test is inconclusive. Therefore, $\mathbf{a}$ could be a local minimum, local maximum, saddle point, or none of the above. One must instead use other information about f, such as its directional derivatives, to determine if f has a maximum, minimum or saddle point at $\mathbf{a}$.

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = 6x^2 + 4xy + 8y^2 - x - 3y.$$

We wish to find any local minima or maxima of this function. First, we compute its gradient,

$$\nabla f = \left[\begin{array}{cc} 12x + 4y - 1 & 4x + 16y - 3 \end{array}\right].$$

To determine where $\nabla f = 0$, we must solve the system of linear equations

$$\begin{aligned} 12x + 4y &= 1, \\ 4x + 16y &= 3. \end{aligned}$$

Using the second equation to obtain $x = (3 - 16y)/4$ and substituting this into the first equation, we obtain $y = 2/11$ and $x = 1/44$. Since the solution of this system is unique, it follows that this is the only critical point of f.

To determine whether this critical point corresponds to a maximum or minimum, we must compute the Hessian H, whose entries are the second partial derivatives of f at $(1/44, 2/11)$. We have

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ 4 & 16 \end{bmatrix}.$$

To determine whether this matrix is positive definite, we first compute its determinant,

$$\det(H) = f_{xx}f_{yy} - f_{xy}^2 = 12(16) - 4(4) = 176.$$

Since the determinant, which is the product of H's two eigenvalues, is positive, it follows that they must both be the same sign.

To determine that sign, we check the diagonal entries of H. If H is positive definite, then the diagonal entries f_{xx} and f_{yy} would both be positive. Therefore, it is sufficient to check f_{xx}; because $\det(H) > 0$, both diagonal entries must the same sign. We have $f_{xx} = 12$, so we conclude that H is positive definite, and that the critical point $(2/11, 1/44)$ is a local *minimum* of f. □

The preceding example describes how the Second Derivatives Test can be performed for a function of two variables:

- If $\det(H) = f_{xx}f_{yy} - f_{xy}^2 > 0$, and $f_{xx} > 0$, then the critical point is a minimum.
- If $\det(H) > 0$ and $f_{xx} < 0$, then the critical point is a maximum.
- If $\det(H) < 0$, then the critical point is a saddle point.
- If $\det(H) = 0$, then the test is inconclusive.

1.8.1 Absolute Extrema

In many applications, it is desirable to know where a function assumes its largest or smallest values, not just among nearby points, but within its entire domain. We say that a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ has an *absolute maximum* at $\mathbf{a}$ if $f(\mathbf{a}) \geq f(\mathbf{x})$ for $\mathbf{x} \in D$, and that f has an *absolute minimum* at $\mathbf{a}$ if $f(\mathbf{a}) \leq f(\mathbf{x})$ for $\mathbf{x} \in D$.

In the single-variable case, it is known, by the Extreme Value Theorem, that if f is continuous on a *closed interval* $[a, b]$, then it has has an absolute maximum and an absolute minimum on $[a, b]$. To find them, it is necessary to check all critical points in $[a, b]$, and the endpoints a and b, as the absolute maximum and absolute minimum must each occur at one of these points.

The generalization of a closed interval to the multivariable case is the notion of a *compact set*. Previously, we defined an open set, and a boundary point. A *closed set* is a set that contains all of its boundary points. A *bounded set* is a set that is contained entirely within a ball $D_r(\mathbf{x}_0)$ for some choice of r and $\mathbf{x}_0$. Finally, a set is *compact* if it is closed and bounded.

We can now state the generalization of the Extreme Value Theorem to the multivariable case. It states that a continuous function on a compact set has an absolute minimum and an absolute maximum. Therefore, given such a compact set D, to find the absolute maximum and minimum, it is sufficient to check the critical points of f in D, and to find the extreme (maximum and minimum) values of f on the boundary. The largest of all of these values is the absolute maximum value, and the smallest is the absolute minimum value.

It should be noted that in cases where D has a simple shape, such as a rectangle, triangle or cube, it is possible to check boundary points by characterizing them using one or more equations, using these equations to eliminate a variable, and then substituting for the eliminated variable in f to obtain a function of one less variable. Then, it is possible to find extreme values on the boundary by solving a maximization or minimization problem in one less dimension.

Example Consider the function $f(x, y) = x^2 + 3y^2 - 4x - 6y$. We will find the absolute maximum and minimum values of this function on the triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 3)$.

First, we look for critical points. We have

$$\nabla f = \begin{bmatrix} 2x - 4 & 6y - 6 \end{bmatrix}.$$

We see that there is only one critical point, at $(x_0, y_0) = (2, 1)$. Because the triangle includes points that satisfy the inequalities $x \geq 0$, $y \geq 0$ and $y \leq 3 - 3x/4$, and the point $(2, 1)$ satisfies all of these inequalities, we conclude that this point lies within the triangle. It is therefore a candidate for an absolute maximum or minimum.

We now check the boundary, by examining each edge of the triangle individually. On the edge between $(0, 0)$ and $(0, 3)$, we have $x = 0$, which yields $f(0, y) = 3y^2 - 6y$. We then have $f_y(0, y) = 6y - 6$, which has a critical point at $y = 1$. Therefore, $(0, 1)$ is also a candidate for an absolute extremum. Similarly, along the edge between $(0, 0)$ and $(4, 0)$, we have $y = 0$, which yields $f(x, 0) = x^2 - 4x$. We then have $f_x(x, 0) = 2x - 4$, which has a critical point at $x = 2$. Therefore, $(2, 0)$ is a candidate for an absolute extremum.

We then check the edge between $(0, 3)$ and $(4, 0)$, along which $y = 3 - 3x/4$. Substituting this into $f(x, y)$ yields the function

$$g(x) = f\left(x, 3 - \frac{3x}{4}\right) = \frac{43}{16}x^2 + 9 - 13x.$$

To determine the critical points of this function, we solve $g'(x) = 0$, which yields $x = 104/43$. Since $y = 3 - 3x/4$ along this edge, the point $(104/43, 51/43)$ is a candidate for an absolute extremum.

Finally, we must include the vertices of the triangle, because they too are boundary points of the triangle, as well as boundary points of the edges along which we attempted to find extrema of single-variable functions. In all, we have seven candidates: the critical point of f, $(2, 1)$, the three critical points found along the edges, $(0, 1)$, $(2, 0)$ and $(104/43, 51/43)$, and the three vertices, $(0, 0)$, $(4, 0)$ and $(0, 3)$. Evaluating $f(x, y)$ at all of these points, we obtain

x	y	f(x,y)
2	1	-7
0	1	-3
2	0	-4
104/43	51/43	$-289/43$
0	0	0
4	0	0
0	3	9

We conclude that the absolute minimum is at $(2, 1)$, and the absolute maximum is at $(0, 3)$. The function is shown on Figure 1.5. □

1.8.2 Relation to Taylor Series

Previously, we learned that when seeking a local minimum or maximum of a function of variables, the Second Derivative Test from single-variable calculus, in which the sign of the second derivative indicated whether a local extremum was a maximum or minimum, generalizes to the Second Derivatives Test, which indicates that a local extremum $\mathbf{x}_0$ is a minimum if the Hessian, the matrix of second partial derivatives, is positive definite at $\mathbf{x}_0$.

We will now use Taylor series to explain why this test is effective. Recall that in single-variable calculus, Taylor's Theorem states that a function $f(x)$

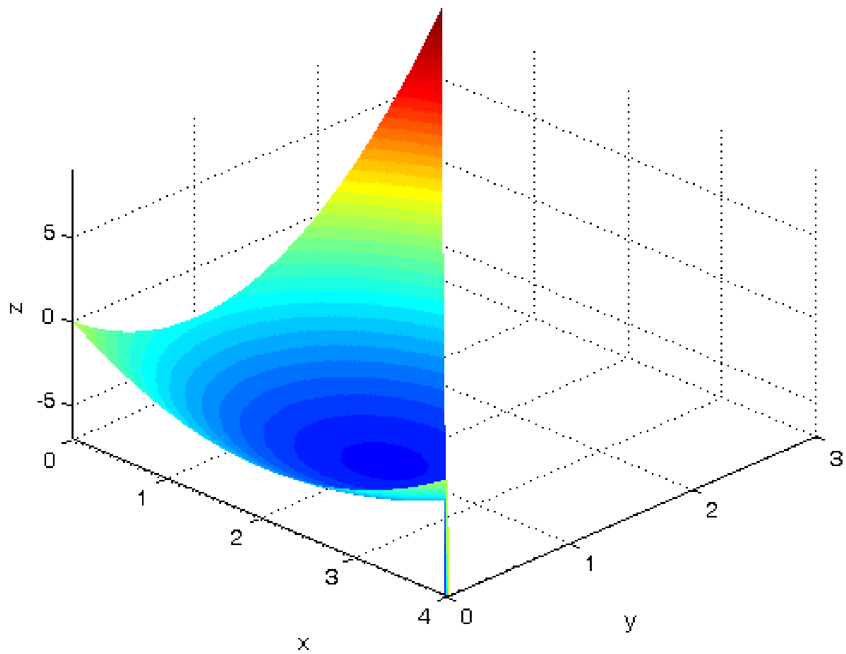

Figure 1.5: The function $f(x,y) = x^2 + 3y^2 - 4x - 6y$ on the triangle with vertices $(0,0)$, $(4,0)$ and $(0,3)$.

with at least three continuous derivatives at x_0 can be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{6}f'''(\xi)(x - x_0)^3,$$

where ξ is between x and x_0. In the multivariable case, Taylor's Theorem states that if $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ has continuous third partial derivatives at $\mathbf{x}_0 \in D$, then

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot H_f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + R_2(\mathbf{x}_0, \mathbf{x}),$$

where $H_f(\mathbf{x}_0)$ is the Hessian, the matrix of second partial derivatives at $\mathbf{x}_0$, defined by

$$H_f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}_0) \\ \vdots & & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}_0) \end{bmatrix},$$

and $R_2(\mathbf{x}_0, \mathbf{x})$ is the *Taylor remainder*, which satisfies

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{R_2(\mathbf{x}_0, \mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0.$$

If we let $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$, then Taylor's Theorem can be rewritten using summations:

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_i^{(0)}) + \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_i^{(0)})(x_j - x_j^{(0)}) + \\ &\quad R_2(\mathbf{x}_0, \mathbf{x}). \end{aligned}$$

Example Let $f(x, y) = x^2y^3 + xy^4$, and let $(x_0, y_0) = (1, -2)$. Then, from partial differentiation of f, we obtain its gradient

$$\nabla f = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 2xy^3 + y^4 & 3x^2y^2 + 4xy^3 \end{bmatrix},$$

and its Hessian,

$$H_f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2y^3 & 6xy^2 + 4y^3 \\ 6xy^2 + 4y^3 & 6x^2y + 12xy^2 \end{bmatrix}.$$

Therefore

$$\nabla f(1, -2) = \begin{bmatrix} 0 & -20 \end{bmatrix}, \quad H_f(1, -2) = \begin{bmatrix} -16 & -8 \\ -8 & 36 \end{bmatrix},$$

and the Taylor expansion of f around $(1, -2)$ is

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle + \\ &\quad \frac{1}{2}\langle x - x_0, y - y_0 \rangle \cdot H_f(x_0, y_0)\langle x - x_0, y - y_0 \rangle + R_2((x_0, y_0), (x, y)) \\ &= 8 + \begin{bmatrix} 0 & -20 \end{bmatrix} \begin{bmatrix} x - 1 \\ y + 2 \end{bmatrix} + \end{aligned}$$

$$\begin{aligned}
& \langle x-1, y+2\rangle \cdot \begin{bmatrix} -16 & -8 \\ -8 & 36 \end{bmatrix} \begin{bmatrix} x-1 \\ y+2 \end{bmatrix} + \\
& R_2((1,-2),(x,y)) \\
= \; & 8 - 20(y+2) - 16(x-1)^2 - 16(x-1)(y+2) + 36(y+2)^2 + \\
& R_2((1,-2),(x,y)).
\end{aligned}$$

The first three terms represent an approximation of $f(x,y)$ by a quadratic function that is valid near the point $(1,-2)$. □

Now, suppose that $\mathbf{x}_0$ is a critical point of $\mathbf{x}$. If this point is to be a local minimum, then we must have $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for $\mathbf{x}$ near $\mathbf{x}_0$. Since $\nabla f(\mathbf{x}_0) = \mathbf{0}$, it follows that we must have

$$(\mathbf{x} - \mathbf{x}_0) \cdot [H_f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)] \geq 0.$$

However, if the Hessian $H_f(\mathbf{x}_0)$ is a *positive definite* matrix, then, by definition, this expression is actually strictly *greater than* zero. Therefore, we are assured that $\mathbf{x}_0$ is a local minimum. In fact, $\mathbf{x}_0$ is a *strict local minimum*, since we can conclude that $f(\mathbf{x}) > f(\mathbf{x}_0)$ for all $\mathbf{x}$ sufficiently near $\mathbf{x}_0$.

1.8.3 Principal Minors

As discussed previously, there are various properties possessed by symmetric positive definite matrices. One other, which provides a relatively straightforward method of checking whether a matrix is positive definite, is to check whether the determinants of its *principal submatrices*, known as *principal minors*, are positive. Given an $n \times n$ matrix A, its principal submatrices are the submatrices consisting of its first k rows and columns, for $k = 1, 2, \ldots, n$. Note that checking these determinants, the principal minors, is equivalent to the test that we have previously described for determining whether a 2×2 matrix is positive definite.

Example Let $f(x,y,z) = x^2 + y^2 + z^2 + xy$. To find any local maxima or minima of this function, we compute its gradient, which is

$$\nabla f(x,y,z) = \begin{bmatrix} 2x+y & 2y+x & 2z \end{bmatrix}.$$

It follows that the only critical point is at $(x_0, y_0, z_0) = (0,0,0)$. To perform the Second Derivatives Test, we compute the Hessian of f, which is

$$H_f(x,y,z) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To determine whether this matrix is positive definite, we can compute the determinants of the principal submatrices of $H_f(0,0,0)$, which are

$$\begin{aligned} [H_f(0,0,0)]_{11} &= 2, \\ [H_f(0,0,0)]_{1:2,1:2} &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \\ [H_f(0,0,0)]_{1:3,1:3} &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

For the principal minors, we have

$$\det([H_f(0,0,0)]_{11}) = 2, \quad \det([H_f(0,0,0)]_{1:2,1:2}) = 2(2) - 1(1) = 3,$$

$$\det([H_f(0,0,0)]_{1:3,1:3}) = 2\det([H_f(0,0,0)]_{1:2,1:2}) = 6.$$

Since all of the principal minors are positive, we conclude that $H_f(0,0,0)$ is positive definite, and therefore the critical point is a minimum of f. □

1.9 Constrained Optimization

Now, we consider the problem of finding the maximum or minimum value of a function $f(\mathbf{x})$, except that the independent variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are subject to one or more *constraints*. These constraints prevent us from using the standard approach for finding extrema, but the ideas behind the standard approach are still useful for developing an approach to the constrained problem.

We assume that the constraints are equations of the form

$$g_i(\mathbf{x}) = 0, \quad i = 1, 2, \ldots, m$$

for given functions $g_i(\mathbf{x})$. That is, we may only consider $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that belong to the intersection of the hypersurfaces (surfaces, when $n = 3$, or curves, when $n = 2$) defined by the g_i, when computing a maximum or minimum value of f. For conciseness, we rewrite these constraints as a vector equation $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with component functions g_i, for $i = 1, 2, \ldots, m$.

By Taylor's theorem, we have, for $\mathbf{x}_0 \in \mathbb{R}^n$ at which $\mathbf{g}$ is differentiable,

$$g(\mathbf{x}) = g(\mathbf{x}_0) + J_{\mathbf{g}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + R_1(\mathbf{x}_0, \mathbf{x}),$$

where $J_{\mathbf{g}}(\mathbf{x}_0)$ is the Jacobian matrix of $\mathbf{g}$ at $\mathbf{x}_0$, consisting of the first partial derivatives of the g_i evaluated at $\mathbf{x}_0$, and $R_1(\mathbf{x}_0, \mathbf{x})$ is the Taylor remainder, which satisfies

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{R_1(\mathbf{x}_0, \mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

It follows that if $\mathbf{u}$ is a vector belonging to *all* of the tangent spaces of the hypersurfaces defined by the g_i, then, because each g_i must remain constant as $\mathbf{x}$ deviates from $\mathbf{x}_0$ in the direction of $\mathbf{u}$, we must have $J_{\mathbf{g}}(\mathbf{x}_0)\mathbf{u} = \mathbf{0}$. In other words, $\nabla g_i(\mathbf{x}_0) \cdot \mathbf{u} = 0$ for $i = 1, 2, \ldots, m$.

Now, suppose that $\mathbf{x}_0$ is a local minimum of $f(\mathbf{x})$, subject to the constraints $\mathbf{g}(\mathbf{x}_0) = \mathbf{0}$. Then, $\mathbf{x}_0$ may not necessarily be a critical point of f, but f may not change along *any* direction from $\mathbf{x}_0$ that satisfies the constraints. Therefore, we must have $\nabla f(\mathbf{x}_0) \cdot \mathbf{u} = 0$ for any vector $\mathbf{u}$ in the intersection of tangent spaces, at $\mathbf{x}_0$, of the hypersurfaces defined by the constraints.

It follows that if $\mathbf{u}$ is any such vector in this tangent plane, and there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \lambda_2 \nabla g_2(\mathbf{x}_0) + \cdots + \lambda_m \nabla g_m(\mathbf{x}_0),$$

then the requirement $\nabla f(\mathbf{x}_0) \cdot \mathbf{u} = 0$ follows directly from the fact that $\nabla g_i(\mathbf{x}_0) \cdot \mathbf{u} = 0$, and therefore $\mathbf{x}_0$ must be a constrained critical point of f. The constants $\lambda_1, \lambda_2, \ldots, \lambda_m$ are called *Lagrange multipliers*.

Example When $m = 1$; that is, when there is only one constraint, the problem of finding a constrained minimum or maximum reduces to finding a point $\mathbf{x}_0$ in the domain of f such that

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0),$$

for a single Lagrange multiplier λ.

Let $f(x, y) = 4x^2 + 9y^2$. The minimum value of this function is at 0, which is attained at $x = y = 0$, but we wish to find the minimum of $f(x, y)$ subject to the constraint $x^2 + y^2 - 2x - 2y = 2$. That is, we must have $g(x, y) = 0$ where $g(x, y) = x^2 + y^2 - 2x - 2y - 2$. To find any points that are candidates for the constrained minimum, we compute the gradients of f and g, which are

$$\nabla f = \begin{bmatrix} 8x & 18y \end{bmatrix},$$
$$\nabla g = \begin{bmatrix} 2x - 2 & 2y - 2 \end{bmatrix}.$$

In order for the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ to be satisfied, we must have, for some choice of λ, x and y,

$$8x = \lambda(2x - 2), \quad 18y = \lambda(2y - 2).$$

From these equations, we obtain

$$x = \frac{\lambda}{\lambda - 4}, \quad y = \frac{\lambda}{\lambda - 9}.$$

Substituting these into the constraint $x^2 + y^2 - 2x - 2y - 2 = 0$ yields the fourth-degree equation

$$4\lambda^4 - 104\lambda^3 + 867\lambda^2 - 2808\lambda + 2592 = 0.$$

This equation has two real solutions,

$$\lambda_1 = \frac{3}{2}, \quad \lambda_2 \approx 13.6.$$

Substituting these values into the above equations for x and y yield the critical points

$$x_1 = -\frac{3}{5}, \quad y1 = -\frac{1}{5}, \quad \lambda_1 = \frac{3}{2},$$

$$x_2 \approx 1.416626, \quad y_2 \approx 2.956124, \quad \lambda_2 \approx 13.6.$$

Substituting the x and y values into $f(x, y)$ yields the minimum value of 9/5 at (x_1, y_1) and the maximum value of approximately 86.675 at (x_2, y_2). □

Example Let $f(x, y, z) = x + y + z$. We wish to find the extremea of this function subject to the constraints $x^2 + y^2 = 1$ and $2x + z = 1$. That is, we must have $g_1(x, y, z) = g_2(x, y, z) = 0$, where $g_1(x, y, z) = x^2 + y^2 - 1$ and $g_2(x, y, z) = 2x + z - 1$. We must find λ_1 and λ_2 such that

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2,$$

or

$$\left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right] = \lambda_1 \left[\begin{array}{ccc} 2x & 2y & 0 \end{array}\right] + \lambda_2 \left[\begin{array}{ccc} 2 & 0 & 1 \end{array}\right].$$

This equation, together with the constraints, yields the system of equations

$$\begin{aligned} 1 &= 2x\lambda_1 + 2\lambda_2 \\ 1 &= 2y\lambda_1 \\ 1 &= \lambda_2 \\ 1 &= x^2 + y^2 \\ 1 &= 2x + z. \end{aligned}$$

From the third equation, $\lambda_2 = 1$, which, by the first equation, yields $2x\lambda_1 = -1$. It follows from the second equation that $x = -y$. This, in conjunction with the fourth equation, yields $(x, y) = (1/\sqrt{2}, -1/\sqrt{2})$ or $(x, y) = (-1/\sqrt{2}, 1/\sqrt{2})$. From the fifth equation, we obtain the two critical points

$$(x_1, y_1, z_1) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1 - \sqrt{2}\right), \quad (x_2, y_2, y_2) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 + \sqrt{2}\right).$$

Substituting these points into f yields $f(x_1, y_1, z_1) = 1-\sqrt{2}$ and $f(x_2, y_2, z_2) = 1+\sqrt{2}$, so we conclude that (x_1, y_1, z_1) is a local minimum of f and (x_2, y_2, z_2) is a local maximum of f, subject to the constraints $g_1(x, y, z) = g_2(x, y, z) = 0$. □

The method of Lagrange multipliers can be used in conjunction with the method of finding unconstrained local maxima and minima in order to find the absolute maximum and minimum of a function on a compact (closed and bounded) set. The basic idea is as follows:

- Find the (unconstrained) critical points of the function, and exclude those that do not belong to the interior of the set.
- Use the method of Lagrange multipliers to find the constrained critical points that lie on the boundary of the set, using equations that characterize the boundary points as constraints. Also, include corners of the boundary, as they represent critical points due to the function, restricted to the boundary, not being differentiable.
- Evaluate the function at all of the constrained and unconstrained critical points. The largest value is the absolute maximum value on the set, and the smallest value is the absolute minimum value on the set.

From a linear algebra point of view, $\nabla f(\mathbf{x}_0)$ must be orthogonal to any vector $\mathbf{u}$ in the *null space* of $J_{\mathbf{g}}(\mathbf{x}_0)$ (that is, the set consisting of any vector $\mathbf{v}$ such that $J_{\mathbf{g}}(\mathbf{x}_0)\mathbf{v} = \mathbf{0}$), and therefore it must lie in the *range* of $J_{\mathbf{g}}(\mathbf{x}_0)^T$, the *transpose* of $J_{\mathbf{g}}(\mathbf{x}_0)$. That is, $\nabla f(\mathbf{x}_0) = J_{\mathbf{g}}(\mathbf{x}_0)^T\mathbf{u}$ for some vector $\mathbf{u}$, meaning that $\nabla f(\mathbf{x}_0)$ must be a *linear combination* of the rows of $J_{\mathbf{g}}(\mathbf{x}_0)$ (the columns of $J_{\mathbf{g}}(\mathbf{x}_0)^T$), which are the gradients of the component functions of $\mathbf{g}$ at $\mathbf{x}_0$.

Another way to view the method of Lagrange multipliers is as a modified *unconstrained* optimization problem. If we define the function $h(\mathbf{x}, \lambda)$ by

$$h(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda \cdot \mathbf{g}(\mathbf{x}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}),$$

then we can find constrained extrema of f by finding unconstrained extrema of h, for

$$\nabla h(\mathbf{x}, \lambda) = \left[\begin{array}{cc} \nabla f(\mathbf{x}) - \lambda \cdot J_{\mathbf{g}}(\mathbf{x}) & -\mathbf{g}(\mathbf{x}) \end{array}\right].$$

Because all components of the gradient must be equal to zero at a critical point (when the gradient exists), the constraints must be satisfied at a critical point of h, and ∇f must be a linear combination of the ∇g_i, so f is only changing along directions that violate the constraints. Therefore, a critical point is a candidate for a constrained maximum or minimum. By the Second Derivatives Test, we can then use the Hessian of h to determine if any constrained extremum is a maximum or minimum.

1.10 Appendix: Linear Algebra Concepts

1.10.1 Matrix Multiplication

As we work with Jacobian matrices for vector-valued functions of several variables, matrix multiplication is a highly relevant operation in multivariable calculus. We have previously defined the product of an $m \times n$ matrix A (that is, A has m rows and n columns) and an $n \times p$ matrix B as the $m \times p$ matrix $C = AB$, where the entry in row i and column j of C is the *dot product* of row i of A and column j of B. This can be written using *sigma notation* as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, p.$$

Note that the number of columns in A must equal the number of rows in B, or the product AB is undefined. Furthermore, in general, even if A and B can be multiplied in either order (that is, if they are square matrices of the same size), AB does not necessarily equal BA. In the special case where the matrix B is actually a column vector $\mathbf{x}$ with n components (that is, $p = 1$), it is useful to be able to recognize the summation

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

as the formula for the ith component of the vector $\mathbf{y} = A\mathbf{x}$.

Example Let A a 3×2 matrix, and B be a 2×2 matrix, whose entries are given by

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 8 \\ 9 & -10 \end{bmatrix}.$$

Then, because the number of columns in A is equal to the number of rows in B, the product $C = AB$ is defined, and equal to the 3×2

$$C = \begin{bmatrix} 1(-7) + (-2)9 & 1(8) + (-2)(-10) \\ (-3)(-7) + 4(9) & (-3)(8) + 4(-10) \\ 5(-7) + (-6)9 & 5(8) + (-6)(-10) \end{bmatrix} = \begin{bmatrix} -25 & 28 \\ 57 & -64 \\ -89 & 100 \end{bmatrix}.$$

Because the number of columns in B is not the same as the number of rows in A, it does not make sense to compute the product BA. □

In multivariable calculus, matrix multiplication most commonly arises when applying the Chain Rule, because the Jacobian matrix of the composition $\mathbf{f} \circ \mathbf{g}$ at point $\mathbf{x}_0$ in the domain of $\mathbf{g}$ is the product of the Jacobian matrix of $\mathbf{f}$, evaluated at $g(\mathbf{x}_0)$, and the Jacobian matrix of $\mathbf{g}$ evaluated at $\mathbf{x}_0$. It follows that the Chain Rule only makes sense when composing functions $\mathbf{f}$ and $\mathbf{g}$ such that the number of *dependent* variables of $\mathbf{g}$ (that is, the number of *rows* in its Jacobian matrix) equals the number of *independent* variables of $\mathbf{f}$ (that is, the number of *columns* in its Jacobian matrix).

Matrix multiplication also arises in Taylor series expansions of multivariable functions, because if $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, then the Taylor expansion of f around $\mathbf{x}_0 \in D$ involves the dot product of $\nabla f(\mathbf{x}_0)$ with the vector $\mathbf{x} - \mathbf{x}_0$, which is a multiplication of a $1 \times n$ matrix with an $n \times 1$ matrix to produce a scalar (by convention, the gradient is written as a *row* vector, while points are written as *column* vectors). Also, such an expansion involves the dot product of $\mathbf{x} - \mathbf{x}_0$ with the product of the Hessian matrix, the matrix of second partial derivatives at $\mathbf{x}_0$, and the vector $\mathbf{x} - \mathbf{x}_0$. Finally, if $\mathbf{g} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function of n variables, then the second term in its Taylor expansion around $\mathbf{x}_0 \in U$ is the product of the Jacobian matrix of $\mathbf{g}$ at $\mathbf{x}_0$ and the vector $\mathbf{x} - \mathbf{x}_0$.

1.10.2 Eigenvalues

Previously, it was mentioned that the eigenvalues of a matrix that is both symmetric, and positive definite, are positive. A scalar λ, which can be real or complex, is an *eigenvalue* of an $n \times n$ matrix A (that is, A has n rows and n columns) if there exists a *nonzero* vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

That is, matrix-vector multiplication of A and $\mathbf{x}$ reduces to a simple scaling of $\mathbf{x}$ by λ. The vector $\mathbf{x}$ is called an *eigenvector* of A corresponding to λ.

The eigenvalues of A are roots of the *characteristic polynomial* $\det(A - \lambda I)$, which is a polynomial of degree n in the variable λ. Therefore, an $n \times n$ matrix A has n eigenvalues, which may repeat. Although the eigenvalues of a matrix may be real or complex, even when the matrix is real, the eigenvalues of a real, symmetric matrix, such as the Hessian of any function with continuous second partial derivatives, are real.

For a general matrix A, $\det(A)$, the determinant of A, is the product of all of the eigenvalues of A. The *trace* of A, denoted by $\operatorname{tr}(A)$, which is defined to be the sum of the diagonal entries of A, is also the sum of the eigenvalues of A. It follows that when A is a 2×2 symmetric matrix, the determinant and trace can be used to easily confirm that the eigenvalues of A are either both positive, both negative, or of opposite signs. This is the basis for the Second Derivatives Test for functions of two variables.

Example Let A be a symmetric 2×2 matrix defined by

$$A = \begin{bmatrix} 4 & -6 \\ -6 & 10 \end{bmatrix}.$$

Then

$$\text{tr}(A) = 4 + 10 = 14, \quad \det(A) = 4(10) - (-6)(-6) = 4.$$

It follows that the product and the sum of A's two eigenvalues are both positive. Because A is symmetric, its eigenvalues are also real. Therefore, they must both also be positive, and we can conclude that A is positive definite.

To actually compute the eigenvalues, we can compute its characteristic polynomial, which is

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 4-\lambda & -6 \\ -6 & 10-\lambda \end{bmatrix}\right) \\ &= (4-\lambda)(10-\lambda) - (-6)(-6) \\ &= \lambda^2 - 14\lambda + 4. \end{aligned}$$

Note that

$$\det(A - \lambda I) = \lambda^2 - \text{tr}(A)\lambda + \det(A),$$

which is true for 2×2 matrices in general. To compute the eigenvalues, we use the quadratic formula to compute the roots of this polynomial, and obtain

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4(4)(1)}}{2(1)} = 7 \pm 3\sqrt{5} \approx 13.708, 0.292.$$

If A represented the Hessian of a function $f(x, y)$ at a point (x_0, y_0), and $\nabla f(x_0, y_0) = \mathbf{0}$, then f would have a local minimum at (x_0, y_0). □

1.10.3 The Transpose, Inner Product and Null Space

The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}\cdot\mathbf{v}$, can also be written as $\mathbf{u}^T\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are both *column* vectors, and $\mathbf{u}^T$ is the *transpose* of $\mathbf{u}$, which converts $\mathbf{u}$ into a *row* vector. In general, the transpose of a matrix A is the matrix A^T whose entries are defined by $[A^T]_{ij} = [A]_{ji}$. That is, in the transpose, the sense of rows and columns are reversed. The dot product is also known as an *inner product*; the *outer product* of two column vectors $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}\mathbf{v}^T$, which is a matrix, whereas the inner product is a scalar.

Given an $m \times n$ matrix A, the *null space* of A is the set $\mathcal{N}(A)$ of all n-vectors such that if $\mathbf{x} \in \mathcal{N}(A)$, then $A\mathbf{x} = \mathbf{0}$. If $\mathbf{x}$ is such a vector, then for any m-vector $\mathbf{v}$, $\mathbf{v}^T(A\mathbf{x}) = \mathbf{v}^T\mathbf{0} = 0$. However, because of two properites of the transpose, $(A^T)^T = A$ and $(AB)^T = B^TA^T$, this inner product can be rewritten as $\mathbf{v}^TA\mathbf{x} = \mathbf{v}^T(A^T)^T\mathbf{x} = (A^T\mathbf{v})^T\mathbf{x}$. It follows that any vector in $\mathcal{N}(A)$ is orthogonal to any vector in the *range* of A^T, denoted by $\mathcal{R}(A^T)$, which is the set of all n-vectors of the form $A^T\mathbf{v}$, where $\mathbf{v}$ is an m-vector. This is the basis for the condition $\nabla f = J_{\mathbf{g}}^T\lambda$ in the method of Lagrange multipliers when there are multiple constraints.

Example Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & -6 \\ 1 & -5 & 10 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 3 & -5 \\ 4 & -6 & 10 \end{bmatrix}.$$

The null space of A, $\mathcal{N}(A)$, consists of all vectors that are multiples of the vector

$$\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix},$$

as it can be verified by matrix-vector multiplication that $A\mathbf{v} = \mathbf{0}$. Now, if we let $\mathbf{w}$ be *any* vector in $\mathbb{R}^3$, and we compute $\mathbf{u} = A^T\mathbf{w}$, then $\mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T\mathbf{u} = 0$, because

$$\mathbf{v}^T\mathbf{u} = \mathbf{v}^TA^T\mathbf{w} = (A\mathbf{v})^T\mathbf{w} = \mathbf{0}^T\mathbf{w} = 0.$$

For example, it can be confirmed directly that $\mathbf{v}$ is orthogonal to any of the columns of A^T. $\square$

Chapter 2

Multiple Integrals

2.1 Double Integrals over Rectangles

In single-variable calculus, the *definite integral* of a function $f(x)$ over an interval $[a, b]$ was defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x,$$

where $\Delta x = (b-a)/n$, and, for each i, $x_{i-1} \leq x_i^* \leq x_i$, where $x_i = a + i\Delta x$.

The purpose of the definite integral is to compute the area of a region with a curved boundary, using the formula for the area of a rectangle. The summation used to define the integral is the sum of the areas of n rectangles, each with width Δx, and height $f(x_i^*)$, for $i = 1, 2, \ldots, n$. By taking the limit as n, the number of rectangles, tends to infinity, we obtain the sum of the areas of infinitely many rectangles of infinitely small width. We define the area of the region bounded by the lines $x = a$, $y = 0$, $x = b$, and the curve $y = f(x)$, to be this limit, if it exists.

Unfortunately, it is too tedious to compute definite integrals using this definition. However, if we define the function $F(x)$ as the definite integral

$$F(x) = \int_a^x f(s)\,ds,$$

then we have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(s)\,ds - \int_a^x f(s)\,ds\right] = \frac{1}{h}\int_x^{x+h} f(s)\,ds.$$

Intuitively, as $h \to 0$, this expression converges to the area of a rectangle of width h and height $f(x)$, divided by the width, which is simply the height, $f(x)$. That is, $F'(x) = f(x)$. This leads to the *Fundamental Theorem of Calculus*, which states that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an *antiderivative* of f; that is, $F' = f$. Therefore, definite integrals are typically evaluated by attempting to undo the differentiation process to find an antiderivative of the *integrand* $f(x)$, and then evaluating this antiderivative at a and b, the *limits* of the integral.

Now, let $f(x, y)$ be a function of two variables. We consider the problem of computing the *volume* of the solid in 3-D space bounded by the surface $z = f(x, y)$, and the planes $x = a$, $x = b$, $y = c$, $y = d$, and $z = 0$, where a, b, c and d are constants. As before, we divide the interval $[a, b]$ into n subintervals of width $\Delta x = (b - a)/n$, and we similarly divide the interval $[c, d]$ into m subintervals of width $\Delta y = (d - c)/m$. For convenience, we also define $x_i = a + i\Delta x$, and $y_j = c + j\Delta y$.

Then, we can approximate the volume V of this solid by the sum of the volumes of mn boxes. The base of each box is a rectangle with dimensions Δx and Δy, and the height is given by $f(x_i^*, y_j^*)$, where, for each i and j, $x_{i-1} \leq x_i^* \leq x_i$ and $y_{j-1} \leq y_j^* \leq y_j$. That is,

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta y\,\Delta x.$$

We then obtain the exact volume of this solid by letting the number of subintervals, n, tend to infinity. The result is the *double integral* of $f(x, y)$ over the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, which is also written as $R = [a, b] \times [c, d]$. The double integral is defined to be

$$V = \int\int_R f(x, y)\,dA = \lim_{m,n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta y\,\Delta x,$$

which is equal to the volume of the given solid. The dA corresponds to the quantity $\Delta A = \Delta x \Delta y$, and emphasizes the fact that the integral is defined to be the limit of the sum of volumes of boxes, each with a base of area ΔA.

To evaluate double integrals of this form, we can proceed as in the single-variable case, by noting that if $f(x_0, y)$, a function of y, is integrable on $[c, d]$

for each $x_0 \in [a, b]$, then we have

$$\begin{aligned}
\int\int_R f(x,y)\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i^*, y_j^*)\,\Delta y\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left[\lim_{m\to\infty} \sum_{j=1}^m f(x_i^*, y_j^*)\Delta y \right] \Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left[\int_c^d f(x_i^*, y)\,dy \right] \Delta x \\
&= \int_a^b \int_c^d f(x,y)\,dy\,dx.
\end{aligned}$$

Similarly, if $f(x, y_0)$, a function of x, is integrable on $[a, b]$ for each $y_0 \in [c, d]$, we also have

$$\int\int_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dy\,dx.$$

This result is known as *Fubini's Theorem*, which states that a double integral of a function $f(x, y)$ can be evaluated as two *iterated* single integrals, provided that f is integrable as a function of either variable when the other variable is held fixed. This is guaranteed if, for instance, $f(x, y)$ is continuous on the entire rectangle R.

That is, we can evaluate a double integral by performing *partial integration* with respect to either variable, x or y, which entails applying the Fundamental Theorem of Calculus to integrate $f(x, y)$ with respect to *only* that variable, while treating the other variable as a constant. The result will be a function of only the other variable, to which the Fundamental Theorem of Calculus can be applied a second time to complete the evaluation of the double integral.

Example Let $R = [0, 1] \times [0, 2]$, and let $f(x, y) = x^2y + xy^3$. We will use Fubini's Theorem to evaluate

$$\int\int_R f(x,y)\,dy\,dx.$$

We have

$$\begin{aligned}
\int\int_R f(x,y)\,dy\,dx &= \int_0^1 \int_0^2 x^2y + xy^3\,dy\,dx \\
&= \int_0^1 \left[\int_0^2 x^2y + xy^3\,dy \right] dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left[\int_0^2 x^2 y \, dy + \int_0^2 x y^3 \, dy \right] dx \\
&= \int_0^1 \left[x^2 \int_0^2 y \, dy + x \int_0^2 y^3 \, dy \right] dx \\
&= \int_0^1 \left[x^2 \left. \frac{y^2}{2} \right|_0^2 + x \left. \frac{y^4}{4} \right|_0^2 \right] dx \\
&= \int_0^1 2x^2 + 4x \, dx \\
&= \left. \left(\frac{2x^3}{3} + 2x^2 \right) \right|_0^1 \\
&= \frac{8}{3}.
\end{aligned}$$

□

In view of Fubini's Theorem, a double integral is often written as

$$\int\int_R f(x,y)\,dA = \int\int_R f(x,y)\,dy\,dx = \int\int_R f(x,y)\,dx\,dy.$$

Example We wish to compute the volume V of the solid bounded by the planes $x = 1$, $x = 4$, $y = 0$, $y = 2$, $z = 0$, and $x + y + z = 8$. The plane that defines the top of this solid is also the graph of the function $z = f(x,y) = 8 - x - y$. It follows that the volume of the solid is given by the double integral

$$V = \int\int_R 8 - x - y\,dA, \quad R = [1,4] \times [0,2].$$

Using Fubini's Theorem, we obtain

$$\begin{aligned}
V &= \int\int_R 8 - x - y \, dA \\
&= \int_1^4 \left[\int_0^2 8 - x - y \, dy \right] dx \\
&= \int_1^4 \left. \left(8y - xy - \frac{y^2}{2} \right) \right|_0^2 dx \\
&= \int_1^4 14 - 2x \, dx \\
&= (14x - x^2)\Big|_1^4
\end{aligned}$$

$$= (56 - 16) - (14 - 1)$$
$$= 27.$$

□

We conclude by noting some useful properties of the double integral, that are direct generalizations of corresponding properties for single integrals:

- *Linearity:* If $f(x, y)$ and $g(x, y)$ are both integrable over R, then
$$\int\int_R [f(x,y) + g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA$$
- *Homogeneity:* If c is a constant, then
$$\int\int_R cf(x,y)\,dA = c\int\int_R f(x,y)\,dA$$
- *Monotonicity:* If $f(x, y) \geq 0$ on R, then
$$\int\int_R f(x,y)\,dA \geq 0.$$
- *Additivity:* If R_1 and R_2 are disjoint rectangles and $Q = R_1 \cup R_2$ is a rectangle, then
$$\int\int_Q f(x,y)\,dA = \int\int_{R_1} f(x,y)\,dA + \int\int_{R_2} f(x,y)\,dA.$$

2.2 Double Integrals over More General Regions

We have learned how to integrate a function $f(x, y)$ of two variables over a rectangle R. However, it is important to be able to integrate such functions over more general regions, in order to be able to compute the volume of a wider variety of solids.

To that end, given a region $D \subset \mathbb{R}^2$, contained within a rectangle R, we define the double integral of $f(x, y)$ over D by

$$\int\int_D f(x,y)\,dA = \int\int_R F(x,y)\,dA$$

where

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \in R, \notin D \end{cases}.$$

It is possible to use Fubini's Theorem to compute integrals over certain types of general regions. We say that a region D is of *type I* if it lies between the graphs of two continuous functions of x, and is also bounded by two vertical lines. Specifically,

$$D = \{(x, y)\,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

To integrate $f(x, y)$ over such a region, we can apply Fubini's Theorem. We let $R = [a, b] \times [c, d]$ be a rectangle that contains D. Then we have

$$\begin{aligned} \int\int_D f(x, y)\, dA &= \int\int_R F(x, y)\, dA \\ &= \int_a^b \int_c^d F(x, y)\, dy\, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy\, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \end{aligned}$$

This is valid because $F(x, y) = 0$ when $y < g_1(x)$ or $y > g_2(x)$, because in these cases, (x, y) lies outside of D. The resulting iterated integral can be evaluated in the same way as iterated integrals over rectangles; the only difference is that when the limits of the inner integral are substituted for y in the antiderivative of $f(x, y)$ with respect to y, the limits are functions of x, rather than constants.

A similar approach can be applied to a *region of type II*, which is bounded on the left and right by continuous functions of y, and bounded above and below by vertical lines. Specifically, D is a region of type II if

$$D = \{(x, y)|h_1(y) \leq x \leq h_2(y), \quad c \leq y \leq d\}.$$

Using Fubini's Theorem, we obtain

$$\int\int_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example We wish to compute the volume of the solid under the plane $x + y + z = 8$, and bounded by the surfaces $y = x$ and $y = x^2$. These surfaces intersect along the lines $x = 0$, $y = 0$ and $x = 1$, $y = 1$. It follows that the volume V of the solid is given by the double integral

$$\int_0^1 \int_{x^2}^{x} 8 - x - y\, dy\, dx.$$

Note that $g_2(x) = x$ is the upper limit of integration, because $x^2 \leq x$ when $0 \leq x \leq 1$. We have

$$\begin{aligned} V &= \int_0^1 \int_{x^2}^x 8 - x - y\,dy\,dx \\ &= \int_0^1 \left(8y - xy - \frac{y^2}{2}\right)\Big|_{x^2}^x \,dx \\ &= \int_0^1 \left(8x - x^2 - \frac{x^2}{2}\right) - \left(8x^2 - x^3 - \frac{x^4}{2}\right)\,dx \\ &= \int_0^1 \frac{x^4}{2} + x^3 - \frac{19x^2}{2} + 8x\,dx \\ &= \left(\frac{x^5}{10} + \frac{x^4}{4} - \frac{19x^3}{6} + 4x^2\right)\Big|_0^1 \\ &= \frac{1}{10} + \frac{1}{4} - \frac{19}{6} + 4 \\ &= \frac{71}{60}. \end{aligned}$$

□

Note that it is sometimes necessary to determine the intersections of surfaces that define a solid, in order to obtain the limits of integration.

To compute the volume of a solid that is bounded above and below (along the z-direction) by two different surfaces, we can add the volume of the solid bounded by the top surface and the plane $z = 0$ to the volume of the solid bounded above by $z = 0$ and below by the lower surface, which is equivalent to subtracting the volume of the solid bounded above by the lower surface and below by $z = 0$.

Example We will compute the volume V of the solid in the first octant bounded by the planes $z = 10 + x + y$, $z = 2 - x - y$, and $x = 0$, as well as the surfaces $y = \sin x$ and $y = \cos x$. As these surfaces intersect along the line $y = \sqrt{2}/2$, $x = \pi/4$, this volume is given by the double integral

$$\begin{aligned} V &= \int_0^{\pi/4} \int_{\sin x}^{\cos x} (10 + x + y) - (2 - x - y)\,dy\,dx \\ &= \int_0^{\pi/4} \int_{\sin x}^{\cos x} 8 + 2x + 2y\,dy\,dx \\ &= \int_0^{\pi/4} \left(8y + 2xy + y^2\right)\Big|_{\sin x}^{\cos x} \,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\pi/4} (2x+8)(\cos x - \sin x) + \cos^2 x - \sin^2 x\,dx \\
&= \int_0^{\pi/4} (2x+8)(\cos x - \sin x) + \cos 2x\,dx \\
&= \left(2x\sin x + 2x\cos x + 6\sin x + 10\cos x + \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/4} \\
&= \frac{\pi\sqrt{2}}{2} + 8\sqrt{2} - \frac{19}{2}.
\end{aligned}
$$

The final anti-differentiation requires integration by parts,

$$\int u\,dv = uv - \int v\,du,$$

with $u = x$ and $dv = (\cos x - \sin x)\,dx$. The function $z = 10 + x + y$ is the "top" plane because for $0 \leq x \leq \pi/4$, $\sin x \leq y \leq cosx$, $10+x+y \geq 2-x-y$. □

By setting the integrand $f(x,y) \equiv 1$ on a region D, and integrating over D, we can obtain $A(D)$, the area of D.

Example We will compute the area of a half-circle by integrating $f(x,y) \equiv 1$ over a region D that is bounded by the planes $z = 0$, $z = 1$, and $y = 0$, and the surface $y = \sqrt{1-x^2}$. This surface intersects the plane $y = 0$ along the lines $y = 0$, $x = 1$ and $y = 0$, $x = -1$. Therefore the area is given by

$$A(D) = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} 1\,dy\,dx = \int_{-1}^{1} y|_0^{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

To evaluate this integral, we use the trigonometric substitution $x = \sin\theta$, for which $dx = \cos\theta\,d\theta$, which yields

$$A(D) = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta = \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2}.$$

□

2.2.1 Changing the Order of Integration

In some cases, a region can be classified as being of *either* type I or type II, and therefore a function can be integrated over the region in two different ways. However, one approach or the other may be impractical, due to the complexity, or even impossibility, of carrying out the anti-differentiation. Therefore, it is important to be able to change the order of integration if necessary.

Example Consider the double integral

$$\int\int_D e^{y^3}\,dA$$

where $D = \{(x,y)\,|\,0 \le x \le 1, \sqrt{x} \le y \le 1\}$. This region is defined as a region of type I, so it is natural to attempt to evaluate the iterated integral

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx.$$

Unfortunately, it is impossible to anti-differentiate e^{y^3} with respect to y. However, the region D is also a region of type II, as it can be redefined as

$$D = \{(x,y)\,|\,0 \le y \le 1, 0 \le x \le y^2\}.$$

We then have

$$\begin{aligned}
\int\int_D e^{y^3}\,dA &= \int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy \\
&= \int_0^1 xe^{y^3}\Big|_0^{y^2}\,dy \\
&= \int_0^1 y^2 e^{y^3}\,dy \\
&= \frac{1}{3}\int_0^1 e^u\,du, \quad u = y^3, \\
&= \frac{1}{3}e^u\Big|_0^1 \\
&= \frac{1}{3}(e-1).
\end{aligned}$$

It should be noted that usually, when changing the order of integration, it is necessary to use the *inverse functions* of the functions that define the curved portions of the boundary, in order to obtain the limits of the integration of the new inner integral.

2.2.2 The Mean Value Theorem for Integrals

It is important to note that all of the properties of double integrals that have been previously discussed, including linearity, homogeneity, monotonicity, and additivity, apply to double integrals over non-rectangular regions as well. One additional property, that is a consequence of monotonicity, is that if $f(x,y) \geq m$ on a region D, and $f(x,y) \leq M$ on D, then

$$mA(D) \leq \int\int_D f(x,y)\,dA \leq MA(D),$$

where, as before, $A(D)$ is the area of D. Furthermore, if f is continuous on D, then, by the *Mean Value Theorem for Double Integrals*, we have

$$\int\int_D f(x,y)\,dA = f(x_0,y_0)A(D),$$

where (x_0, y_0) is some point in D. This is a generalization of the Mean Value Theorem for Integrals, which is closely related to the Mean Value Theorem for derivatives.

Example Consider the double integral

$$\int\int_D e^y\,dA$$

where D is the triangle defined by $D = \{(x,y)\,|\,0 \leq x \leq 1, 0 \leq y \leq 4x\}$. The area of this triangle is given by $A(D) = \frac{1}{2}bh$, where b, the base, is 1 and h, the height, is 4, which yields $A(D) = 2$. Because $1 \leq e^y \leq e^4$ when $0 \leq y \leq 4$, it follows that

$$2 \leq \int\int_D e^y\,dA \leq 2e^4 \approx 109.2.$$

The exact value is $\frac{1}{4}(e^4 - 5) \approx 12.4$, which is between the above lower and upper bounds. □

2.3 Double Integrals in Polar Coordinates

We have learned how to integrate functions of two variables, x and y, over various regions that have a simple form. The variables x and y correspond to *Cartesian coordinates* that are normally used to describe points in 2-D space. However, a region that may not be of type I or type II, when described using Cartesian coordinates, may be of one of these types if it is instead described using *polar coordinates* r and θ.

We recall that polar coordinates are related to Cartesian coordinates by the equations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

or, alternatively,

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

In order to integrate a function over a region defined using polar coordinates, we must derive the double integral in these coordinates, as was previously done in Cartesian coordinates.

Let a solid be bounded by the surface $z = f(r, \theta)$, as well as the surfaces $r = a$, $r = b$, $\theta = \alpha$ and $\theta = \beta$, which define a *polar rectangle*. To compute the volume of this solid, we can approximate it by several solids for which the volume can easily be computed. This is accomplished by dividing the polar rectangle into several smaller polar rectangles of dimensions Δr and $\Delta\theta$. The height of each solid is obtained from the value of the function at a point in the polar rectangle.

Specifically, we divide the interval $[a, b]$ into n subintervals of width $\Delta r = (b - a)/n$. Each subinterval is of the form $[r_{i-1}, r_i]$, where $r_i = a + i\Delta r$, for $i = 1, 2, \ldots, n$. Similarly, $[\alpha, \beta]$ is divided into m subintervals of width $\Delta\theta = (\beta - \alpha)/m$, and each subinterval is of the form $[\theta_{j-1}, \theta_j]$, where $\theta_j = \alpha + j\Delta\theta$. Then, the volume V of the solid is approximated by

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{2} f(r_i^*, \theta_j^*)(r_i^2 - r_{i-1}^2)\Delta\theta,$$

where, for each i, $r_{i-1} \leq r_i^* \leq r_i$, and for each j, $\theta_{j-1} \leq \theta_j^* \leq \theta_j$.

The quantity $\frac{1}{2}\Delta r^2 \Delta\theta$ is the area of a polar rectangle, for it is not truly a rectangle, but rather the difference between two circular sectors with angle $\Delta\theta$ and radii r_{i-1} and r_i. However, from

$$\frac{1}{2}(r_i^2 - r_{i-1}^2) = \frac{1}{2}(r_{i-1} + r_i)(r_i - r_{i-1}) = \frac{1}{2}(r_{i-1} + r_i)\Delta r,$$

we see that as $m, n \to \infty$, this approximation of the volume converges to the exact volume, which is given by the double integral

$$V = \int_\alpha^\beta \int_a^b f(r, \theta)\, r\, dr\, d\theta.$$

Note the extra factor of r in the integrand, which is the limit as $n \to \infty$ of $(r_{i-1} + r_i)/2$.

If the base of the solid can be represented by a polar region of type I,

$$D = \{(r, \theta) \,|\, \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then the volume V of the solid defined by the surface $z = f(r, \theta)$ and the surfaces that define D is given by the iterated integral

$$V = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

As before, if $f(r, \theta) \equiv 1$, then this integral yields $A(D)$, the area of D.

Example To evaluate the double integral

$$\int\int_D x + y\, dA,$$

where $D = \{(x, y) \,|\, 1 \leq x^2 + y^2 \leq 4, x \leq 0\}$, we convert the integrand, and the description of D, to polar coordinates. We then have

$$\int\int_D r\cos\theta + r\sin\theta\, dA$$

where $D = \{(r, \theta) \,|\, 1 \leq r \leq 2, \pi/2 \leq \theta \leq 3\pi/2\}$. This simplifies the integral considerably, because D can be described as a polar rectangle. We then have

$$\begin{aligned} \int\int_D x + y\, dA &= \int_{\pi/2}^{3\pi/2} \int_1^2 (r\cos\theta + r\sin\theta) r\, dr\, d\theta \\ &= \int_{\pi/2}^{3\pi/2} \int_1^2 r^2(\cos\theta + \sin\theta)\, dr\, d\theta \\ &= \int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta) \left. \frac{r^3}{3} \right|_1^2 d\theta \\ &= \frac{7}{3} \int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta)\, d\theta \\ &= \frac{7}{3} \left. (\sin\theta - \cos\theta) \right|_{\pi/2}^{3\pi/2} \\ &= \frac{7}{3}[(-1 - 0) - (1 - 0)] \\ &= -\frac{14}{3}. \end{aligned}$$

□

Example To compute the volume of the solid in the first octant bounded below by the cone $z = \sqrt{x^2 + y^2}$, and above by the sphere $x^2 + y^2 + z^2 = 8$, as well as the planes $y = x$ and $y = 0$, we first rewrite the equations of the bounding surfaces in polar coordinates. The solid is bounded below by the cone $z = r$, above by the sphere $r^2 + z^2 = 8$, and the surfaces $\theta = 0$ and $\theta = \pi/4$, since the solid lies in the first octant. The surfaces that bound the solid above and below intersect when $2r^2 = 8$, or $r = 2$. It follows that the volume is given by

$$\begin{aligned} V &= \int_0^{\pi/4} \int_0^2 [\sqrt{8 - r^2} - r] r \, dr \, d\theta \\ &= \int_0^{\pi/4} \int_0^2 r\sqrt{8 - r^2} \, dr \, d\theta - \int_0^{\pi/4} \int_0^2 r^2 \, dr \, d\theta \\ &= -\frac{1}{2} \int_0^{\pi/4} \int_8^4 u^{1/2} \, du \, d\theta - \int_0^{\pi/4} \left. \frac{r^3}{3} \right|_0^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \left. \frac{2}{3} u^{3/2} \right|_4^8 d\theta - \int_0^{\pi/4} \frac{8}{3} \, d\theta \\ &= \frac{1}{3} \int_0^{\pi/4} [16\sqrt{2} - 8] \, d\theta - \frac{2\pi}{3} \\ &= \frac{4\pi}{3} [\sqrt{2} - 1]. \end{aligned}$$

In the third step, the substitution $u = 8 - r^2$ is used. Then, the limits of integration are interchanged in order to reverse the sign of the integral. □

Example The double integral

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x, y) \, dy \, dx$$

can be converted to polar coordinates by converting the equation that describes the top boundary of the domain of integration, $y = \sqrt{1 - x^2}$, into a polar equation. We substitute $x = r\cos\theta$ and $y = r\sin\theta$ into this equation to obtain

$$r \sin\theta = \sqrt{1 - \cos^2\theta}.$$

Squaring both sides yields $r^2 \sin^2\theta = 1 - \cos^2\theta$, and, in view of the identity $\cos^2\theta + \sin^2\theta = 1$, we obtain the polar equation $r = 1$. Because the bottom boundary, $y = 0$, corresponds to the rays $\theta = 0$ and $\theta = \pi$, the integral can be expressed in polar coordinates as

$$\int_0^{\pi} \int_0^1 f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

□

Example We evaluate the double integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

by converting to polar coordinates By completing the square, we obtain $2x - x^2 = 1 - (x-1)^2$. It follows that the region D over which the integral is to be evaluated,

$$D = \{(x,y) \,|\, 0 \leq x \leq 2, 0 \leq y \leq \sqrt{2x - x^2}\},$$

has its top boundary defined by the equation $y = \sqrt{2x - x^2}$, or

$$(x-1)^2 + y^2 = 1.$$

That is, the top boundary is the upper half of the circle with radius 1 and center $(1, 0)$. In polar coordinates, the equation of the top boundary becomes

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1,$$

or, upon expanding and simplifying,

$$r = 2\cos\theta.$$

The region D is contained between the rays $\theta = 0$ and $\theta = \pi/2$. It follows that in polar coordinates, D is defined by

$$D = \{(r,\theta) \,|\, 0 \leq \theta \leq \pi/2, 0 \leq r \leq 2\cos\theta\}.$$

The lower limit $r = 0$ is obtained from the fact that D contains the origin. We thus obtain the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta.$$

The integrand of the original integral is r, but the additional factor of r required by the change to polar coordinates yields an integrand of r^2.

Evaluating this integral, we obtain

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx &= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= \int_0^{\pi/2} \left.\frac{r^3}{3}\right|_0^{2\cos\theta} d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos^2\theta\cos\theta\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos\theta\,d\theta - \frac{8}{3}\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta \\
&= \frac{8}{3}\sin\theta|_0^{\pi/2} - \frac{8}{3}\int_0^1 u^2\,du \\
&= \frac{8}{3}(1) - \frac{8}{3}\left.\frac{u^3}{3}\right|_0^1 \\
&= \frac{16}{9}.
\end{aligned}$$

□

2.4 Triple Integrals

The integral of a function of three variables over a region $D \subset \mathbb{R}^3$ can be defined in a similar way as the double integral. Let D be the box defined by

$$D = \{(x,y,z)\,|\,a \le x \le b, c \le y \le d, r \le z \le s\}.$$

Then, as with the double integral, we divide $[a,b]$ into n subintervals of width $\Delta x = (b-a)/n$, with endpoints $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. Similarly, we divide $[c,d]$ into m subintervals of width $\Delta y = (d-c)/m$, with endpoints $[y_{j-1}, y_j]$, for $j = 1, 2, \ldots, m$, and divide $[r,s]$ into ℓ subintervals of width $\Delta z = (s-r)/\ell$, with endpoints $[z_{k-1}, z_k]$ for $k = 1, 2, \ldots, \ell$.

Then, we can define the *triple integral* of a function $f(x,y,z)$ over D by

$$\int\int\int_D f(x,y,z)\,dV = \lim_{m,n,\ell\to\infty} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^{\ell} f(x_i^*, y_j^*, z_k^*)\,\Delta V,$$

where $\Delta V = \Delta x \Delta y \Delta z$. As with double integrals, the practical method of evaluating a triple integral is as an iterated integral, such as

$$\int\int\int_D f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

By Fubini's Theorem, which generalizes to three dimensions or more, the order of integration can be rearranged when f is continuous on D.

A triple integral over a more general region can be defined in the same way as with double integrals. If E is a bounded subset of $\mathbb{R}^3$, that is contained within a box B, then we can define

$$\int\int\int_E f(x,y,z)\,dV = \int\int\int_B F(x,y,z)\,dV,$$

where

$$F(x,y,z) = \begin{cases} f(x,y,z) & (x,y,z) \in E, \\ 0 & (x,y,z) \notin E \end{cases}.$$

All of the properties previously associated with the double integral, such as linearity and additivity, generalize to the triple integral as well.

Just as regions were classified as type I or type II for double integrals, they can be classified for the purpose of setting up triple integrals. A solid region E is said to be of *type 1* if it lies between the graphs of two continuous functions of x and y that are defined on a two-dimensional region D. Specifically,

$$E = \{(x,y,z) \,|\, (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}.$$

Then, an integral of a function $f(x,y,z)$ over E can be evaluated as

$$\int\int\int_E f(x,y,z)\,dV = \int\int_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA,$$

where the double integral over D can be evaluated in a manner that is appropriate for the type of D.

For example, if D is of type I, then

$$E = \{(x,y,z) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x,y) \leq z \leq u_2(x,y)\},$$

and therefore

$$\int\int\int_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

On the other hand, if E is of *type 2*, then it has a definition of the form

$$E = \{(x, y, z) \,|\, (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}.$$

That is, E lies between the graphs of two continuous functions of y and z that are defined on a two-dimensional region D. Finally, if E is a region of *type 3*, then it lies between the graphs of two continuous functions of x and z. That is,

$$E = \{(x, y, z) \,|\, (x, z) \in D, u_1(y, z) \leq y \leq u_2(y, z)\}.$$

If more than one type applies to a given region E, then the order of evaluation can be determined by which ordering leads to the integrands that are most easily anti-differentiated within each single integral that arises.

Example Let E be a solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. We wish to integrate the function $f(x, y, z) = xz$ over this tetrahedron. From the given bounding planes, we see that the tetrahedron is bounded below by the plane $z = 0$ and above by the plane $z = 1 - x - y$. Therefore, we surmise that E can be viewed as a solid of type 1. This requires finding a region D in the xy-plane such that E is bounded by $z = 0$ and $z = 1 - x - y$ on D.

We first note that these planes intersect along the line $x + y = 1$. It follows that the base of E is a 2-D region D that can be described by the inequalities $x \geq 0$, $y \geq 0$, and $x + y \leq 1$. This region is of type I or type II, so we choose type I and obtain the description

$$D = \{(x, y) \,|\, 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}.$$

Therefore, we can integrate $f(x, y, z)$ over E as follows:

$$\begin{aligned}
\int\int\int_E xz\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xz\,dz\,dy\,dx \\
&= \int_0^1 x \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dz \\
&= \int_0^1 x \int_0^{1-x} \left.\frac{z^2}{2}\right|_0^{1-x-y} dy\,dx \\
&= \frac{1}{2}\int_0^1 x \int_0^{1-x} (1 - x - y)^2\,dy\,dx \\
&= \frac{1}{2}\int_0^1 x \left.\left(-\frac{(1 - x - y)^3}{3}\right)\right|_0^{1-x} dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6}\int_0^1 x(1-x)^3\,dx \\
&= \frac{1}{6}\int_0^1 (1-u)u^3\,du, \quad u = 1-x \\
&= \frac{1}{6}\int_0^1 u^3 - u^4\,du \\
&= \frac{1}{6}\left(\frac{u^4}{4} - \frac{u^5}{5}\right)\Big|_0^1 \\
&= \frac{1}{6}\left(\frac{1}{4} - \frac{1}{5}\right) \\
&= \frac{1}{120}.
\end{aligned}$$

□

Example We will compute the volume of the solid E bounded by the surfaces $y = x$, $y = x^2$, $z = x$, and $z = 0$. Because E is bounded by two surfaces that define z as a function of x and y, we view E as a solid of type 1. It is bounded by the graphs of the functions $z = 0$ and $z = x$ that are defined on a region D in the xy-plane. This region is bounded by the curves $y = x$ and $y = x^2$. Because these curves intersect when $x = 0$ and $x = 1$, we can describe D as a region of type I:

$$D = \{(x, y) \,|\, 0 \leq x \leq 1, x^2 \leq y \leq x\}.$$

It follows that the volume of E is given by the iterated integral

$$\begin{aligned}
\int\int\int_E 1\,dV &= \int_0^1 \int_{x^2}^x \int_0^x 1\,dz\,dy\,dx \\
&= \int_0^1 \int_{x^2}^x x\,dy\,dx \\
&= \int_0^1 x \int_{x^2}^x 1\,dy\,dx \\
&= \int_0^1 x(x - x^2)\,dx \\
&= \int_0^1 x^2 - x^3\,dx \\
&= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\
&= \frac{1}{12}.
\end{aligned}$$

□

Example We evaluate the triple integral

$$\int\int\int_E x\,dV$$

where E is the solid bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$. The paraboloid and the plane intersect when $y^2 + z^2 = 1$. It follows that the right boundary of the solid E is the unit disk $y^2 + z^2 \leq 1$, contained within the plane $x = 4$. Because the paraboloid serves as the "left" boundary of E, we can define E by the inequalities

$$E = \{(x, y, z) \,|\, y^2 + z^2 \leq 1, 4(y^2 + z^2) \leq x \leq 4\}.$$

Therefore, the triple integral can be written as an iterated integral

$$\int\int\int_E x\,dV = \int\int_D \left[\int_{4(y^2+z^2)}^4 x\,dx\right] dA,$$

where D is the unit disk in the yz-plane, $y^2 + z^2 = 1$. If we convert y and z to polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we can rewrite this integral as

$$\int\int\int_E x\,dV = \int_0^{2\pi}\int_0^1\int_{4r^2}^4 xr\,dx\,dr\,d\theta.$$

Evaluating this integral, we obtain

$$\begin{aligned}\int\int\int_E x\,dV &= \int_0^{2\pi}\int_0^1\int_{4r^2}^4 xr\,dx\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r\frac{x^2}{2}\bigg|_{4r^2}^4\,dr\,d\theta \\ &= 8\int_0^{2\pi}\int_0^1 r(1-r^4)\,dr\,d\theta \\ &= 8\int_0^{2\pi}\int_0^1 r - r^5\,dr\,d\theta \\ &= 8\int_0^{2\pi} \frac{r^2}{2} - \frac{r^6}{6}\bigg|_0^1\,d\theta \\ &= 8\int_0^{2\pi}\frac{1}{3}\,d\theta \\ &= \frac{16\pi}{3}.\end{aligned}$$

□

2.5 Applications of Double and Triple Integrals

We now explore various applications of double and triple integrals arising from physics. When an object has constant density ρ, then it is known that its mass m is equal to ρV, where V is its volume. Now, suppose that a flat plate, also known as a *lamina*, has a non-uniform density $\rho(x, y)$, for $(x, y) \in D$, where D defines the shape of the lamina. Then, its mass is given by

$$m = \int\int_D \rho(x, y)\, dA.$$

Similarly, if E is a solid region in 3-D space, and $\rho(x, y, z)$ is the density of the solid at the point $(x, y, z) \in E$, then the mass of the solid is given by

$$m = \int\int\int_E \rho(x, y, z)\, dV.$$

We see that just as the integral allows simple "product" formulas for area and volume to be applied to more general problems, it allows similar formulas for quantities such as mass to be generalized as well.

The *center of mass*, also known as the *center of gravity*, of an object is the point at which the object behaves as if its entire mass is concentrated at that point. If the object is one- or two-dimensional, the center of mass is the point at which the object can be balanced horizontally (like a see-saw with riders at either end, in the one-dimensional case).

For a lamina with its shape defined by a bounded region $D \subset \mathbb{R}^2$, and with density given by$\rho(x, y)$, its center of mass $(\bar{x}, \bar{y})$ is located at

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m},$$

where M_x and M_y are the *moments* of the lamina about the x-axis and y-axis, respectively. These are given by

$$M_x = \int\int_D y\rho(x, y)\, dA, \quad M_y = \int\int_D x\rho(x, y)\, dA.$$

These integrals are obtained from the formula for the moment of a point mass about an axis, which is given by the product of the mass and the distance from the axis.

Similarly, the moments about the xy-, yz- and xz-planes, M_{xy}, M_{yz}, and M_{xz}, of a solid $E \subset \mathbb{R}^3$ with density $\rho(x, y, z)$ are given by

$$M_{xy} = \int\int\int_E z\rho(x, y, z)\, dV,$$

$$\begin{aligned} M_{yz} &= \int\int\int_E x\rho(x,y,z)\,dV, \\ M_{xz} &= \int\int\int_E y\rho(x,y,z)\,dV. \end{aligned}$$

It follows that its center of mass $(\bar{x},\bar{y},\bar{z})$ is located at

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

As in the 2-D case, each moment is defined using the distance of each point of E from the coordinate plane about which the moment is being computed.

The *moment of interia*, or *second moment*, of an object about an axis gives an indication of the object's tendency to rotate about that axis. For a lamina defined by a region $D \subset \mathbb{R}^2$ with density function $\rho(x,y)$, its moments of inertia about the x-axis and y-axis, I_x and I_y respectively, are given by

$$I_x = \int\int_D y^2\rho(x,y)\,dA, \quad I_y = \int\int_D x^2\rho(x,y)\,dA.$$

On the other hand, for a solid defined by a region $E \subset \mathbb{R}^3$ with density $\rho(x,y,z)$, its moments of inertia about the coordinate axes are defined by

$$I_x = \int\int\int_E (y^2+z^2)\rho(x,y,z)\,dV, \quad I_y = \int\int\int_E (x^2+z^2)\rho(x,y,z)\,dV,$$

$$I_z = \int\int\int_E (x^2+y^2)\rho(x,y,z)\,dV.$$

The moment I_z is also called the *polar moment of interia*, or the *moment of interia about the origin*, when E reduces to a lamina with density $\rho(x,y)$.

2.6 Triple Integrals in Cylindrical Coordinates

We have seen that in some cases, it is convenient to evaluate double integrals by converting Cartesian coordinates (x,y) to polar coordinates (r,θ). The same is true of triple integrals. When this is the case, Cartesian coordinates (x,y,z) are converted to *cylindrical coordinates* (r,θ,z).

The relationships between (x,y) and (r,θ) are exactly the same as in polar coordinates, and the z coordinate is unchanged.

Example The point $(x, y, z) = (-3, 3, 4)$ can be converted to cylindrical coordinates (r, θ, z) using the relationships from polar coordinates,

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

These relationships yield

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}, \quad \tan\theta = -1.$$

Since $x = -3 < 0$, we have $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$. We conclude that the cylindrical coordinates of the point $(-3, 3, 4)$ are $(3\sqrt{2}, 3\pi/4, 4)$. □

Furthermore, just as conversion to polar coordinates in double integrals introduces a factor of r in the integrand, conversion to cylindrical coordinates in triple integrals also introduces a factor of r.

Example We evaluate the triple integral

$$\int\int\int_E f(x, y, z)\, dV,$$

where E is the solid bounded below by the paraboloid $z = x^2 + y^2$, above by the plane $z = 4$, and the planes $y = 0$ and $y = 2$. This integral can be evaluated as an iterated integral

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} f(x, y, z)\, dz\, dy\, dx,$$

but if we instead describe the region using cylindrical coordinates, we find that the solid is bounded below by the paraboloid $z = r^2$, above by the plane $z = 4$, and contained within the polar "box" $0 \leq r \leq 2$, $0 \leq \theta \leq \pi$. We can therefore evaluate the iterated integral

$$\int_0^2 \int_0^\pi \int_{r^2}^4 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, d\theta\, dr,$$

that has much simpler limits. □

Example We use cylindrical coordinates to evaluate the triple integral

$$\int\int\int_E x\, dV$$

where E is the solid bounded by the planes $z = 0$ and $z = x + y + 5$, and the cylindrical shells $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$. In cylindrical coordinates, E

is bounded by the planes $z = 0$ and $z = r(\cos\theta + \sin\theta) + 5$, and the cylinders $r = 2$ and $r = 3$. It follows that the integral can be written as the iterated integral

$$\int\int\int_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r(\cos\theta+\sin\theta)+5} (r\cos\theta)r\,dz\,dr\,d\theta.$$

Evaluating this integral, we obtain

$$\begin{aligned}
\int\int\int_E x\,dV &= \int_0^{2\pi}\cos\theta\int_2^3 r^2\int_0^{r(\cos\theta+\sin\theta)+5} dz\,dr\,d\theta \\
&= \int_0^{2\pi}\cos\theta\int_2^3 [r^3(\cos\theta+\sin\theta)+5r^2]\,dr\,d\theta \\
&= \int_0^{2\pi}\cos\theta(\cos\theta+\sin\theta)\int_2^3 r^3\,dr\,d\theta + \int_0^{2\pi}\cos\theta\int_2^3 5r^2\,dr\,d\theta \\
&= \int_0^{2\pi}\cos\theta(\cos\theta+\sin\theta)\left.\frac{r^4}{4}\right|_2^3 d\theta + 5\int_0^{2\pi}\cos\theta\left.\frac{r^3}{3}\right|_2^3 d\theta \\
&= \frac{65}{4}\int_0^{2\pi}\cos^2\theta+\sin\theta\cos\theta\,d\theta + \frac{95}{3}\int_0^{2\pi}\cos\theta\,d\theta \\
&= \frac{65}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)+\frac{1}{2}\sin 2\theta\,d\theta + \frac{95}{3}\sin\theta|_0^{2\pi} \\
&= \frac{65}{4}\left.\left[\frac{1}{2}\theta+\frac{1}{2}\sin 2\theta)-\frac{1}{4}\cos 2\theta\right]\right|_0^{2\pi} d\theta \\
&= \frac{65\pi}{4}.
\end{aligned}$$

□

2.7 Triple Integrals in Spherical Coordinates

Another approach to evaluating triple integrals, that is especially useful when integrating over regions that are at least partially defined using spheres, is to use *spherical coordinates*. Consider a point (x, y, z) that lies on a sphere of radius ρ. Then we know that $x^2 + y^2 + z^2 = \rho^2$. Furthermore, the points $(0, 0, 0)$, $(0, 0, z)$ and (x, y, z) form a right triangle with hypotenuse ρ and legs $|z|$ and $\sqrt{\rho^2 - z^2}$.

If we denote by ϕ the angle adjacent to the leg of length $|z|$, then ϕ can be interpreted as an *angle of inclination* of the point (x, y, z). The angle $\phi = 0$ corresonds to the "north pole" of the sphere, while $\phi = \pi/2$ corresponds to the "equator", and $\phi = \pi$ corresponds to the "south pole". By right triangle trigonometry, we have

$$z = \rho \cos \phi.$$

It follows that $x^2 + y^2 = \rho^2 \sin^2 \phi$. If we define the angle θ to have the same meaning as in polar coordinates, then we have

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta.$$

We define the *spherical coordinates* of (x, y, z) to be (ρ, θ, ϕ).

Example To convert the point $(x, y, z) = (1, \sqrt{3}, -4)$ to spherical coordinates, we first compute

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + (\sqrt{3})^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}.$$

Next, we use the relation $\tan \theta = y/x$, and the fact that $x = 1 > 0$, to obtain

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \sqrt{3} = \frac{\pi}{3}.$$

Finally, to obtain ϕ, we use the relation $z = \rho \cos \phi$, which yields

$$\phi = \cos^{-1} \frac{z}{\rho} = \cos^{-1} \left(-\frac{4}{2\sqrt{5}} \right) \approx 2.6779 \, \text{radians}.$$

□

To evaluate integrals in spherical coordinates, it is important to note that the volume of a "spherical box" of dimensions Δr, $\Delta\theta$ and $\Delta\phi$, as $\Delta\rho, \Delta\theta, \Delta\phi \to 0$, converges to the infinitesimal

$$\rho^2 \sin \phi \, dr \, d\theta \, d\phi,$$

where (ρ, θ, ϕ) denotes the location of the box in the limit. Therefore, the integral of a function $f(x, y, z)$ over a solid E, when evaluated in spherical coordinates, becomes

$$\int\int\int_E f(x, y, z) \, dV = \int\int\int_E f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

Example We wish to compute the volume of the solid E in the first octant bounded below by the plane $z = 0$ and the hemisphere $x^2 + y^2 + z^2 = 9$, bounded above by the hemisphere $x^2 + y^2 + z^2 = 16$, and the planes $y = 0$ and $y = x$. This would be highly inconvenient to attempt to evaluate in

Cartesian coordinates; determining the limits in z alone requires breaking up the integral with respect to z. However, in spherical coordinates, the solid E is determined by the inequalities

$$3 \leq \rho \leq 4, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq \phi \leq \frac{\pi}{2}.$$

That is, the solid is actually a "spherical rectangle". It follows that the volume V is given by the iterated integral

$$\begin{aligned} V &= \int_0^{\pi/2} \int_0^{\pi/4} \int_3^4 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{\pi}{4} \int_0^{\pi/2} \int_3^4 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{\pi}{4} \int_0^{\pi/2} \sin\phi \int_3^4 \rho^2 \, d\rho \, d\theta \, d\phi \\ &= \frac{\pi}{4} \int_0^{\pi/2} \sin\phi \left. \frac{\rho^3}{3} \right|_3^4 d\theta \, d\phi \\ &= \frac{\pi}{4} \frac{37}{3} \int_0^{\pi/2} \sin\phi \, d\theta \, d\phi \\ &= -\frac{\pi}{4} \frac{37}{3} \cos\phi|_0^{\pi/2} \\ &= \frac{37\pi}{12}. \end{aligned}$$

□

Example We use spherical coordinates to evaluate the triple integral

$$\int\int\int_H (x^2 + y^2) \, dV,$$

where H is the solid that is bounded below by the xy-plane, and bounded above by the sphere $x^2 + y^2 + z^2 = 1$. In spherical coordinates, H is defined by the inequalities

$$H = \{(\rho, \theta, \phi) \,|\, 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2\}.$$

As the integrand $x^2 + y^2$ is equal to $(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 = \rho^2\sin^2\phi$ in spherical coordinates, we have

$$\int\int\int_H (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin^2\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Evaluating this integral, we obtain

$$\begin{aligned}
\int\int\int_H (x^2+y^2)\,dV &= \int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi \int_0^1 \rho^4\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi \left.\frac{\rho^5}{5}\right|_0^1 d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin^3\phi\,d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin^2\phi\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} (1-\cos^2\phi)\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\,d\phi\,d\theta - \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi} (-\cos\phi)|_0^{\pi/2}\,d\theta - \frac{1}{5}\int_0^{2\pi}\int_0^1 u^2\,du\,d\theta \\
&= \frac{1}{5}\int_0^{2\pi} 1\,d\theta - \frac{1}{5}\int_0^{2\pi}\left.\frac{u^3}{3}\right|_0^1 d\theta \\
&= \frac{1}{5}\left(2\pi - \frac{2\pi}{3}\right) \\
&= \frac{4\pi}{15}.
\end{aligned}$$

□

2.8 Change of Variables in Multiple Integrals

Recall that in single-variable calculus, if the integral

$$\int_a^b f(u)\,du$$

is evaluated by making a change of variable $u = g(x)$, such that the interval $\alpha \leq x \leq \beta$ is mapped by g to the interval $a \leq u \leq b$, then

$$\int_a^b f(u)\,du = \int_\alpha^\beta f(g(x))g'(x)\,dx.$$

The appearance of the factor $g'(x)$ in the integrand is due to the fact that if we divide $[a,b]$ into n subintervals $[u_{i-1}, u_i]$ of equal width $\Delta u =$

$(b-a)/n$, and if we divide $[\alpha, \beta]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (\beta - \alpha)/n$, then

$$\Delta u = u_i - u_{i-1} = g(x_i) - g(x_{i-1}) = g'(x_i^*)\Delta x,$$

where $x_{i-1} \leq x_i^* \leq x_i$. We will now generalize this change of variable to multiple integrals.

For simplicity, suppose that we wish to evaluate the double integral

$$\int\int_D f(x,y)\, dA$$

by making a change of variable

$$x = g(u,v), \quad y = h(u,v), \quad a \leq u \leq b, \quad c \leq v \leq d.$$

We divide the interval $[a,b]$ into n subintervals $[u_{i-1}, u_i]$ of equal width $\Delta u = (b-a)/n$, and we divide $[c,d]$ into m subintervals $[v_{i-1}, v_i]$ of equal width $\Delta v = (d-c)/m$. Then, the rectangle $[u_{i-1}, u_i] \times [v_{i-1}, v_i]$ is approximately mapped by g and h into a parallelogram with adjacent sides

$$\mathbf{r}_u = \langle g(u_i, v_{i-1}) - g(u_{i-1}, v_{i-1}), h(u_i, v_{i-1}) - h(u_{i-1}, v_{i-1})\rangle,$$

$$\mathbf{r}_v = \langle g(u_{i-1}, v_i) - g(u_{i-1}, v_{i-1}), h(u_{i-1}, v_i) - h(u_{i-1}, v_{i-1})\rangle.$$

By the Mean Value Theorem, we have

$$\begin{aligned} \mathbf{r}_u &\approx \langle g_u(u_{i-1}, v_{i-1}), h_u(u_{i-1}, v_{i-1})\rangle \Delta u, \\ \mathbf{r}_v &\approx \langle g_v(u_{i-1}, v_{i-1}), h_v(u_{i-1}, v_{i-1})\rangle \Delta v. \end{aligned}$$

The area of this parallelogram is given by

$$|\mathbf{r}_u \times \mathbf{r}_v| = \left| \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} \right| \Delta u \Delta v.$$

It follows that

$$\int\int_D f(x,y)\, dx\, dy = \int\int_{\tilde{D}} f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv,$$

where $\tilde{D} = [a,b] \times [c,d]$ is the domain of g and h, and

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

is the *Jacobian* of the transformation from (u, v) to (x, y). It is also the *determinant* of the Jacobian matrix of the vector-valued function that maps (u, v) to (x, y).

Example Let D be the parallelogram with vertices $(0, 0)$, $(2, 4)$, $(6, 1)$, and $(8, 5)$. To integrate a function $f(x, y)$ over D, we can use a change of variable $(x, y) = (g(u, v), h(u, v))$ that maps a rectangle to this parallelogram, and then integrate over the rectangle.

Using the vertices, we find that the equations of the edges are

$$-x + 6y = 0, \quad -x + 6y = 22, \quad 2x - y = 0, \quad 2x - y = 11.$$

Therefore, if we define the new variables u and v by the equations

$$u = -x + 6y, \quad v = 2x - y,$$

then, for $(x, y) \in D$, we have (u, v) belonging to the rectangle $0 \leq u \leq 22$, $0 \leq v \leq 11$.

To rewrite an integral over D in terms of u and v, it is much easier to express the original variables in terms of the new variables than the other way around. Therefore, we need to solve the equations defining u and v for x and y. From the equation for u, we have $x = 6y - u$. Substituting into the equation for v, we obtain $v = 2(6y - u) - y$, which yields $y = h(u, v) = \frac{1}{11}(2u + v)$. Subtituting this into the equation for u yields $x = g(u, v) = \frac{1}{11}(u + 6v)$.

The Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{11^2}[1(1) - 6(2)] = -\frac{1}{11}.$$

We conclude that

$$\int\int_D f(x, y)\, dx\, dy = \frac{1}{11}\int\int_{\tilde{D}} f(g(u, v), h(u, v))\, du\, dv.$$

□

In general, when integrating a function $f(x_1, x_2, \ldots, x_n)$ over a region $D \subset \mathbb{R}^n$, if the integral is evaluated using a change of variable $(x_1, x_2, \ldots, x_n) = \mathbf{g}(u_1, u_2, \ldots, u_n)$ that maps a region $E \subset \mathbb{R}^n$ to D, then

$$\int_D f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \int_E (f \circ \mathbf{g})(u_1, \ldots, u_n) |\det(J_{\mathbf{g}}(u_1, \ldots, u_n))|\, du_1 \cdots du_n,$$

where

$$J_{\mathbf{g}}(u_1, u_2, \ldots, u_n) = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix}$$

is the Jacobian matrix of $\mathbf{g}$ and $\det(J_{\mathbf{g}}(u_1, u_2, \ldots, u_n))$ is its determinant, which is simply referred to as the Jacobian of the transformation $\mathbf{g}$.

Example Consider the transformation from spherical to Cartesian coordinates,

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Then, the Jacobian matrix of this transformation is

$$\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}.$$

It follows that the Jacobian of this transformation is given by the determinant of this matrix,

$$\begin{aligned} \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} &= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \\ & \quad \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &= \cos\phi[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta] - \\ & \quad \rho\sin\phi[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta] \\ &= -\rho^2\cos^2\phi\sin\phi - \rho^2\sin^2\phi\sin\phi \\ &= -\rho^2\sin\phi. \end{aligned}$$

The absolute value of the Jacobian is the factor that must be included in the integrand when converting a triple integral from Cartesian to spherical coordinates. □

Example We evaluate the double integral

$$\int\int_R (x^2 - xy + y^2)\, dA,$$

where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$, using the change of variables

$$x = \sqrt{2}u - \sqrt{2/3}v, \quad y = \sqrt{2}u + \sqrt{2/3}v.$$

First, we compute the Jacobian of the change of variables,

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\left[\begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array}\right]\right) = \det\left(\left[\begin{array}{cc} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{array}\right]\right) = \sqrt{2}\sqrt{2/3}+\sqrt{2}\sqrt{2/3} = \frac{4}{\sqrt{3}}.$$

Next, we need to define the region R in terms of u and v. Rewriting the equation $x^2-xy+y^2 = 2$ in terms of u and v yields the equation $2u^2+2v^2 = 2$. It follows that the change of variables maps the region $\tilde{R}$ to R, where $\tilde{R}$ is the unit disk. If we then use polar coordinates $u = r\cos\theta$ and $v = r\sin\theta$, we have

$$\int\int_R (x^2-xy+y^2)\,dA = \int\int_{\tilde{R}} (2u^2+2v^2)\frac{4}{\sqrt{3}}\,du\,dv = \frac{4}{\sqrt{3}}\int_0^{2\pi}\int_0^1 (2r^2)r\,dr\,d\theta.$$

Evaluating this integral, we obtain

$$\begin{aligned} \int\int_R (x^2 - xy + y^2)\,dA &= \frac{8}{\sqrt{3}}\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta \\ &= \frac{8}{\sqrt{3}}\int_0^{2\pi} \left.\frac{r^4}{4}\right|_0^1 d\theta \\ &= \frac{2}{\sqrt{3}}\int_0^{2\pi} 1\,d\theta \\ &= \frac{4\pi}{\sqrt{3}}. \end{aligned}$$

□

Example We wish to use an appropriate change of variable to evaluate the double integral

$$\int\int_R (x+y)e^{x^2-y^2}\,dA,$$

where R is the rectangle enclosed by the lines $x-y = 0$, $x-y = 2$, $x+y = 0$ and $x+y = 3$. If we define $u = x+y$ and $v = x-y$, then R is mapped by this change of variables to the rectangle

$$\tilde{R} = \{(u,v)\,|\,0 \le u \le 3, 0 \le v \le 2\}.$$

Solving for x and y in terms of u and v, we obtain

$$x = \frac{1}{2}(u+v), \quad y = \frac{1}{2}(u-v).$$

It follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\frac{1}{2} = -\frac{1}{2}$$

and the integral becomes

$$\int\int_R (x+y)e^{x^2-y^2}\,dA = \int\int_R (x+y)e^{(x+y)(x-y)}\,dA = \int_0^3\int_0^2 ue^{uv}\left|-\frac{1}{2}\right|\,dv\,du.$$

Evaluating this integral, we obtain

$$\begin{aligned}
\int\int_R (x+y)e^{x^2-y^2}\,dA &= \frac{1}{2}\int_0^3\int_0^2 ue^{uv}\,dv\,du \\
&= \frac{1}{2}\int_0^3 e^{uv}\big|_0^2\,du \\
&= \frac{1}{2}\int_0^3 [e^{2u}-1]\,du \\
&= \frac{1}{2}\left[\frac{e^{2u}}{2}-u\right]\Bigg|_0^3 \\
&= \frac{1}{2}\left(\frac{e^6}{2}-3-\frac{1}{2}\right) \\
&= \frac{1}{4}(e^6-7).
\end{aligned}$$

□

Chapter 3

Vector Calculus

3.1 Vector Fields

To this point, we have mostly worked with scalar-valued functions of several variables, in the interest of computing quantities such as the maximum or minimum value of a function, or the volume or center of mass of a solid. Now, we will study applications involving *vector-valued* functions of several variables. The difficulty of visualizing such functions leads to the notion of a *vector field*.

A function $\mathbf{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a function that assigns to each point $\mathbf{x} \in U$ a vector

$$\mathbf{F}(\mathbf{x}) = \langle F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x}) \rangle$$

in $\mathbb{R}^n$. The functions $F_1, F_2, \ldots, F_n$ are the *component functions*, or *component scalar fields*, of $\mathbf{F}$. For our purposes, $n = 2$ or 3. To visualize a vector field, one can plot the vector $F(\mathbf{x})$ at any given point $\mathbf{x}$, using the component functions to obtain the components of the vector to be plotted at each point.

The following are certain vector fields of interest in applications:

- Given a fluid, for example, a *velocity field* is a vector field $\mathbf{V}(x, y, z)$ that indicates the velocity of the fluid at each point (x, y, z). When plotting a velocity field, the speed of the fluid at each point is indicated by the length of the vector plotted at that point, and the direction of the fluid at that point is indicated by the direction of the vector.

  A curve $\mathbf{c}(t)$ is said to be a *flow line*, or *streamline*, of a velocity field $\mathbf{V}$ if, for each value of the parameter t,

  $$\mathbf{c}'(t) = \mathbf{V}(\mathbf{c}(t)).$$

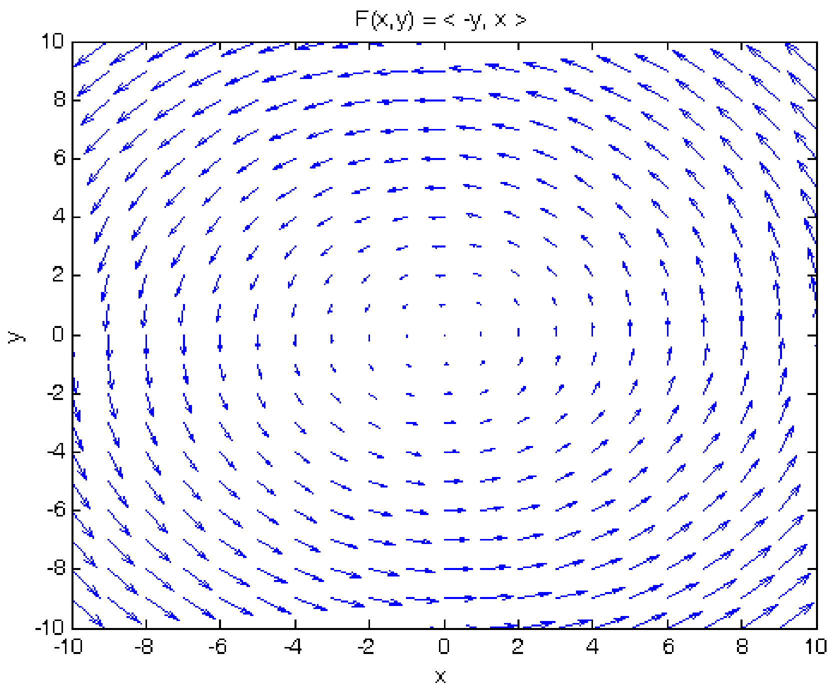

Figure 3.1: The vector field $\mathbf{V}(x, y) = \langle -y, x\rangle$

That is, at each point along the curve, its tangent vector coincides with $\mathbf{V}$. A flow line can be approximated by first choosing an initial point $\mathbf{x}_0 = \mathbf{c}(t_0)$, then using the value of $\mathbf{V}$ at that point to approximate a second point $\mathbf{x}_1 = \mathbf{c}(t_1)$ as follows:

$$\frac{\mathbf{x}_1 - \mathbf{x}_0}{t_1 - t_0} = \frac{\mathbf{c}(t_1) - \mathbf{c}(t_0)}{t_1 - t_0} \approx \mathbf{V}(\mathbf{c}(t_0)) \quad \Longrightarrow \quad \mathbf{x}_1 \approx \mathbf{x}_0 + (t_1 - t_0)\mathbf{V}(\mathbf{x}_0).$$

This can be continued to obtain the locations of any number of points along the flow line. The closer the times $t_0, t_1, \ldots$ are to one another, the more accurate the approximate flow line will be.

- Consider two objects with mass m and M, with the object of mass M located at the origin, and the vector field $\mathbf{F}$ defined by

$$\mathbf{F}(\mathbf{r}) = -\frac{mMG}{\|\mathbf{r}\|^3}\mathbf{r},$$

where $\mathbf{r}$ is a position vector of the object of mass m, and G is the gravitational constant. This vector field indicates the gravitational force exerted by the object at the origin on the object at position $\mathbf{r}$, and is therefore an example of a *gravitational field*.

- Suppose an electric charge Q is located at the origin, and a charge q is located at the point with position vector $\mathbf{x}$. Then the electric force exerted by the first charge on the second is given by the vector field

$$F(\mathbf{x}) = \frac{\varepsilon q Q}{\|\mathbf{x}\|^3}\mathbf{x},$$

where ε is a constant. This field, and the gravitational field described above, are both examples of *force fields*.

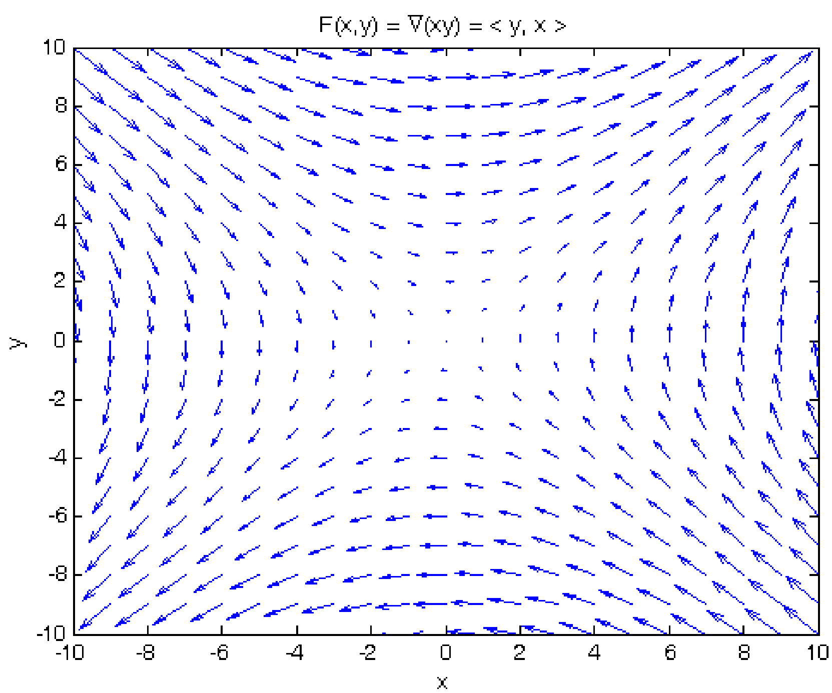

Figure 3.2: The conservative vector field $\mathbf{F}(x, y) = \langle y, x\rangle$

- A vector field $\mathbf{F}$ is said to be *conservative* if $\mathbf{F} = \nabla f$ for some function f. We also say that $\mathbf{F}$ is a *gradient field*, and f is a *potential function* for

F. When we discuss line integrals, we will learn the physical meaning of a conservative vector field.

In upcoming sections we will learn how to integrate vector fields, as well as the physical interpretations of such integrals.

Example Consider the velocity field $\mathbf{V}(x, y) = \langle -y, x\rangle$. It is shown in Figure 3.1. It can be seen from the figure that the flow lines of this velocity field are circles centered at the origin. □

Example The vector field $\mathbf{F}(x, y) = \langle y, x\rangle$ is conservative, because $\mathbf{F} = \nabla f$, where $f(x, y) = xy$. The field is shown in Figure 3.2. It should be noted that conservative vector fields are also called *irrotational*; a fluid whose velocity field is conservative has no *vorticity*. □

3.2 Line Integrals

Recall from single-variable calclus that if a constant force F is applied to an object to move it along a straight line from $x = a$ to $x = b$, then the amount of work done is the force times the distance, $W = F(b - a)$. More generally, if the force is not constant, but is instead dependent on x so that the amount of force applied when the object is at the point x is given by $F(x)$, then the work done is given by the integral

$$W = \int_a^b F(x)\,dx.$$

This result is obtained by applying the "basic" formula for work along each of n subintervals of width $\Delta x = (b - a)/n$, and taking the limit as $\Delta x \to 0$.

Now, suppose that a force is applied to an object to move it along a path traced by a curve C, instead of moving it along a straight line. If the amount of force that is being applied to the object at any point $\mathbf{p}$ on the curve C is given by the value of a function $F(\mathbf{p})$, then the work can be approximated by, as before, applying the "basic" formula for work to each of n line segments that approximate the curve and have lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. The work done on the ith segment is approximately $F(\mathbf{p}_i^*)\Delta s_i$, where $\mathbf{p}_i^*$ is any point on the segment. By taking the limit as $\max \Delta s_i \to 0$, we obtain the *line integral*

$$W = \int_C F(\mathbf{p})\,ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^n F(\mathbf{p}_i^*)\,\Delta s_i,$$

provided that this limit exists.

In order to actually evaluate a line integral, it is necessary to express the curve C in terms of parametric equations. For concreteness, we assume that C is a plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b.$$

Then, if we divide $[a, b]$ into subintervals of width $\Delta t = (b-a)/n$, with endpoints $[t_{i-1}, t_i]$ where $t_i = a + i\Delta t$, we can approximate C by n line segments with endpoints $(x(t_{i-1}), y(t_{i-1}))$ and $(x(t_i), y(t_i))$, for $i = 1, 2, \ldots, n$. From the Pythagorean Theorem, it follows that the ith segment has length

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\,\Delta t,$$

where $\Delta x_i = x(t_i) - x(t_{i-1})$ and $\Delta y_i = y(t_i) - y(t_{i-1})$. Letting $\Delta t \to 0$, we obtain

$$\int_C F(\mathbf{p})\,ds = \int_a^b F(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

We recall that if $F(x, y) \equiv 1$, then this integral yields the arc length of the curve C.

Example (Stewart, Section 13.2, Exercise 8) To evaluate the line integral

$$\int_C x^2 z\,ds$$

where C is the line segment from $(0, 6, -1)$ to $(4, 1, 5)$, we first need parametric equations for the line segment. Using the vector between the endpoints,

$$\mathbf{v} = \langle 4 - 0, 1 - 6, 5 - (-1)\rangle = \langle 4, -5, 6\rangle,$$

we obtain the parametric equations

$$x = 4t, \quad y = 6 - 5t, \quad z = -1 + 6t, \quad 0 \leq t \leq 1.$$

It follows that

$$\begin{aligned}
\int_C x^2 z\,ds &= \int_0^1 (x(t))^2 z(t)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt \\
&= \int_0^1 (4t)^2(6t-1)\sqrt{4^2 + (-5)^2 + 6^2}\,dt \\
&= \int_0^1 16t^2(6t-1)\sqrt{77}\,dt
\end{aligned}$$

$$\begin{aligned} &= 16\sqrt{77}\int_0^1 6t^3 - t^2\,dt \\ &= 16\sqrt{77}\left(6\frac{t^4}{4} - \frac{t^3}{3}\right)\Big|_0^1 \\ &= 16\sqrt{77}\left(\frac{3}{2} - \frac{1}{3}\right) \\ &= \frac{56\sqrt{77}}{3}. \end{aligned}$$

□

Example (Stewart, Section 13.2, Exercise 10) We evaluate the line integral

$$\int_C (2x + 9z)\,ds$$

where C is defined by the parametric equations

$$x = t, \quad y = t^2, \quad z = t^3, \quad 0 \leq t \leq 1.$$

We have

$$\begin{aligned} \int_C (2x + 9z)\,ds &= \int_0^1 (2x(t) + 9z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt \\ &= \int_0^1 (2t + 9t^3)\sqrt{1^2 + (2t)^2 + (3t^2)^2}\,dt \\ &= \int_0^1 (2t + 9t^3)\sqrt{1 + 4t^2 + 9t^4}\,dt \\ &= \frac{1}{4}\int_1^{14} u^{1/2}\,du, \quad u = 1 + 4t^2 + 9t^4 \\ &= \frac{1}{4}\,\frac{2}{3}u^{3/2}\Big|_1^{14} \\ &= \frac{1}{6}(14^{3/2} - 1). \end{aligned}$$

□

Although we have introduced line integrals in the context of computing work, this approach can be used to integrate any function along a curve. For example, to compute the mass of a wire that is shaped like a plane curve C, where the density of the wire is given by a function $\rho(x, y)$ defined at each point (x, y) on C, we can evaluate the line integral

$$m = \int_C \rho(x, y)\, ds.$$

It follows that the center of mass of the wire is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds, \quad \bar{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds.$$

Now, suppose that a *vector-valued* force $\mathbf{F}$ is applied to an object to move it along the path traced by a plane curve C. If we approximate the curve by line segments, as before, the work done along the ith segment is approximately given by

$$W_i = \mathbf{F}(\mathbf{p}_i^*) \cdot [\mathbf{T}(\mathbf{p}_i^*)\Delta s_i]$$

where $\mathbf{p}_i^*$ is a point on the segment, and $\mathbf{T}(\mathbf{p}_i^*)$ is the *unit* tangent vector to the curve at this point. That is, $\mathbf{F} \cdot \mathbf{T} = \|\mathbf{F}\| \cos\theta$ is the amount of force that is applied to the object at each point on the curve, where θ is the angle between $\mathbf{F}$ and the direction of the curve, which is indicated by $\mathbf{T}$. In the limit as $\max \Delta s_i \to 0$, we obtain the *line integral of* $\mathbf{F}$ *along* C,

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

If the curve C is parametrized by the the vector equation $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where $a \leq t \leq b$, then the tangent vector is parametrized by

$$\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|,$$

and, as before, $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \|\mathbf{r}'(t)\|\, dt$. It follows that

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \|\mathbf{r}'(t)\|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The last form of the line integral is merely an abbreviation that is used for convenience. As with line integrals of scalar-valued functions, the parametric representation of the curve is necessary for actual evaluation of a line integral.

Example (Stewart, Section 13.2, Exercise 20) We evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = \langle z, y, -x\rangle$ and C is the curve defined by the parametric vector equation

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle t, \sin t, \cos t\rangle, \quad 0 \leq t \leq \pi.$$

We have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^\pi \langle z(t), y(t), -x(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt \\
&= \int_0^\pi \langle \cos t, \sin t, -t\rangle \cdot \langle 1, \cos t, -\sin t\rangle\,dt \\
&= \int_0^\pi [\cos t + \sin t \cos t + t \sin t]\,dt \\
&= \int_0^\pi \cos t\,dt + \int_0^\pi \sin t \cos t\,dt + \int_0^\pi t \sin t\,dt \\
&= \sin t|_0^\pi + \frac{1}{2}\sin^2 t\Big|_0^\pi - t\cos t|_0^\pi + \int_0^\pi \cos t\,dt \\
&= \pi.
\end{aligned}$$

□

If we write $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$, where P and Q are the component functions of $\mathbf{F}$, then we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b \langle P(x(t), y(t)), Q(x(t), y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\,dt \\
&= \int_a^b P(x(t), y(t))x'(t)\,dt + \int_a^b Q(x(t), y(t))y'(t)\,dt.
\end{aligned}$$

When the curve is approximated by n line segments, as before, the difference in the x-coordinates of each segment is, by the Mean Value Theorem,

$$\Delta x_i = x(t_i) - x(t_{i-1}) \approx x'(t_i^*)\,\Delta t,$$

where $t_{i-1} \leq t_i^* \leq t_i$. For this reason, we write

$$\int_a^b P(x(t), y(t))x'(t)\,dt = \int_C P\,dx,$$

$$\int_a^b Q(x(t), y(t))y'(t)\,dt = \int_C Q\,dy,$$

and conclude

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_C P\,dx + Q\,dy.$$

These line integrals of scalar-valued functions can be evaluated individually to obtain the line integral of the vector field $\mathbf{F}$ over C. However, it is important to note that unlike line integrals with respect to the arc length s, the value of line integrals with respect to x or y (or z, in 3-D) depends on the *orientation* of C. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well. We denote by $-C$ the curve C with its orientation reversed. We then have

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = -\int_{-C} \mathbf{F}\cdot\,d\mathbf{r},$$

and

$$\int_C P\,dx = -\int_{-C} P\,dx, \quad \int_C Q\,dy = -\int_{-C} Q\,dy.$$

All of this discussion generalizes to space curves (that is, curves in 3-D) in a straightforward manner, as illustrated in the examples.

Example (Stewart, Section 13.2, Exercise 6) Let $\mathbf{F}(x, y) = \langle \sin x, \cos y\rangle$ and let C be the curve that is the top half of the circle $x^2 + y^2 = 1$, traversed counterclockwise from $(1, 0)$ to $(-1, 0)$, and the line segment from $(-1, 0)$ to $(-2, 3)$. To evaluate the line integral

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \sin x\,dx + \cos y\,dy,$$

we consider the integrals over the semicircle, denoted by C_1, and the line segment, denoted by C_2, separately. We then have

$$\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy.$$

For the semicircle, we use the parametric equations

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \pi.$$

This yields

$$\int_{C_1} \sin x\,dx + \cos y\,dy \quad = \quad \int_0^\pi \sin(\cos t)(-\sin t)\,dt + \cos(\sin t)\cos t\,dt$$

$$\begin{aligned} &= -\cos(\cos t)|_0^\pi + \sin(\sin t)|_0^\pi \\ &= -\cos(-1) + \cos(1) \\ &= 0. \end{aligned}$$

For the line segment, we use the parametric equations

$$x = -1 - t, \quad y = 3t, \quad 0 \leq t \leq 1.$$

This yields

$$\begin{aligned} \int_{C_2} \sin x\,dx + \cos y\,dy &= \int_0^1 \sin(-1-t)(-1)\,dt + \cos(3t)(3)\,dt \\ &= -\cos(-1-t)|_0^1 + \sin(3t)|_0^1 \\ &= -\cos(-2) + \cos(-1) + \sin(3) - \sin(0) \\ &= -\cos(2) + \cos(1) + \sin(3). \end{aligned}$$

We conclude

$$\int_C \sin x\,dx + \cos y\,dy = \cos(1) - \cos(2) + \sin(3).$$

In evaluating these integrals, we have taken advantage of the rule

$$\int_a^b f'(g(t))g'(t)\,dt = f(g(b)) - f(g(a)),$$

from the Fundamental Theorem of Calculus and the Chain Rule. However, this shortcut can only be applied when an integral involves only one of the independent variables. □

Example (Stewart, Section 13.2, Exercise 12) We evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle z, x, y\rangle,$$

and C is defined by the parametric equations

$$x = t^2, \quad y = t^3, \quad z = t^2, \quad 0 \leq t \leq 1.$$

We have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C P\,dx + Q\,dy + R\,dz \\
&= \int_0^1 z(t)x'(t)\,dt + x(t)y'(t)\,dt + y(t)z'(t)\,dt \\
&= \int_0^1 t^2(2t)\,dt + t^2(3t^2)\,dt + t^3(2t)\,dt \\
&= \int_0^1 2t^3\,dt + 3t^4\,dt + 2t^4\,dt \\
&= \int_0^1 (5t^4 + 2t^3)\,dt \\
&= \left(5\frac{t^5}{5} + 2\frac{t^4}{4}\right)\bigg|_0^1 \\
&= \frac{3}{2}.
\end{aligned}$$

□

3.3 The Fundamental Theorem for Line Integrals

We have learned that the line integral of a vector field $\mathbf{F}$ over a curve piecewise smooth C, that is parameterized by a vector-valued function $\mathbf{r}(t)$, $a \leq t \leq b$, is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt.$$

Now, suppose that $\mathbf{F}$ continuous, and is a *conservative* vector field; that is, $\mathbf{F} = \nabla f$ for some scalar-valued function f. Then, by the Chain Rule, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b \frac{d}{dt}[(f \circ \mathbf{r})(t)]\,dt = (f \circ \mathbf{r})(t)|_a^b = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

This is the *Fundamental Theorem of Line Integrals*, which is a generalization of the Fundamental Theorem of Calculus.

If the curve C is a *closed* curve; that is, the initial and terminal points of C are the same, then $\mathbf{r}(b) = \mathbf{r}(a)$, which yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0.$$

If we decompose C into two curves C_1 and C_2, and use the fact that the sign of the line integral of a vector field over a curve depends on the orientation of the curve, then we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = 0.$$

That is,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{-C_2} \mathbf{F}\cdot d\mathbf{r}.$$

However, C_1 and $-C_2$ have the same initial and terminal points. It follows that if $\mathbf{F}$ is conservative within an open, *connected* domain D (so that any two points in D can be connected by a path that lies within D), then the line integral of $\mathbf{F}$ is *independent of path* in D; that is, the value of the line integral of $\mathbf{F}$ over a path C depends only on its initial and terminal points.

The *converse* of this statement is also true: if the line integral of a vector field $\mathbf{F}$ is independent of path within an open, connected domain D, then $\mathbf{F}$ is a conservative vector field on D. To see this, we consider the two-variable case and let D be a region in $\mathbb{R}^2$. Furthermore, we let $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. We choose an arbitrary point $(a,b) \in D$, and define

$$f(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}.$$

Since this line integral is independent of path, we can define $f(x,y)$ using any path between (a,b) and (x,y) that we choose, knowing that its value at (x,y) will be the same in any case.

By choosing a path that ends with a horizontal line segment from (x_1,y) to (x,y) contained entirely in D, parametrized by $x = t$, $y = y$, for $x_1 \leq t \leq x$, we can show that

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left[\int_{(a,b)}^{(x_1,y)} \mathbf{F}\cdot d\mathbf{r}\right] + \frac{\partial}{\partial x}\left[\int_{(x_1,y)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}\right] \\
&= 0 + \frac{\partial}{\partial x}\left[\int_{x_1}^{x} P(x(t),y)x'(t)\,dt + Q(x(t),y)y'(t)\,dt\right] \\
&= \frac{\partial}{\partial x}\left[\int_{x_1}^{x} P(t,y)\,dt + 0\right] \\
&= P(x,y).
\end{aligned}$$

Using a similar argument, we can show that $\partial f/\partial y = Q$. We have thus shown that $\mathbf{F}$ is conservative, and conclude that $\mathbf{F}$ *is a conservative vector field if and only if its line integral is independent of path.*

However, in order to use the Fundamental Theorem of Line Integrals to evaluate the line integral of a conservative vector field, it is necessary to obtain the function f such that $\nabla f = F$. Furthermore, the theorem cannot be applied to a vector field that is not conservative, so we need to be able to confirm that a given vector field is conservative before we proceed.

Continuing to restrict ourselves to the two-variable case, suppose that $\mathbf{F} = \langle P, Q\rangle$ is a conservative vector field defined on a domain D, and that P and Q have continuous first partial derivatives. Then, we have

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q,$$

for some function f. It follows that

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

However, by Clairaut's Theorem, these mixed second partial derivatives of f are equal, so it follows that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

if $\mathbf{F} = \langle P, Q\rangle$ is conservative.

If the domain D is *simply connected*, meaning that *any* region enclosed by a closed curve in D contains only points in D (informally, D has "no holes"), then the converse is true: if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

in D, then $\mathbf{F} = \langle P, Q\rangle$ is a conservative vector field. Similarly, if $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field defined on a simply connected domain $D \subseteq \mathbb{R}^3$, and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then $\mathbf{F}$ is conservative.

It remains to be able to find the function f such that $\nabla f = \mathbf{F}$ for a given vector field $\mathbf{F} = \langle P, Q\rangle$ that is known to be conservative. The general technique is as follows:

- Integrate $P(x, y)$ with respect to x to obtain

$$f(x, y) = f_1(x, y) + g(y),$$

where $f_1(x, y)$ is obtained by anti-differentiation of $P(x, y)$, and $g(y)$ is an unknown function that plays the role of the constant of integration, since $f(x, y)$ is obtained by anti-differentiating with respect to x.

- Differentiate f with respect to y to obtain

$$\frac{\partial}{\partial y}[f_1(x, y)] + g'(y) = Q(x, y),$$

and solve for $g'(y)$.

- Integrate $g'(y)$ with respect to y to complete the definition of $f(x, y)$, up to a constant of integration.

A similar procedure can be used for a vector field defined on $\mathbb{R}^3$, except that the function g depends on both y and z, and differentiation with respect to both y and z is needed to completely define the function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

Example (Stewart, Section 13.3, Exercise 14) Let

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = \langle 2xz + y^2, 2xy, x^2 + 3z^2\rangle.$$

To confirm that $\mathbf{F}$ is conservative, we check the appropriate first partial derivatives of P, Q and R:

$$P_y = 2y = Q_x, \quad P_z = 2x = R_x, \quad Q_z = 0 = R_y.$$

Now, to find a function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$, which must satisfy $f_x = P$, we integrate $P(x, y, z)$ with respect to x and obtain

$$f(x, y, z) = x^2 z + y^2 x + g(y, z).$$

Differentiating with respect to y and z yields the equations

$$f_y(x, y, z) = 2xy + g_y(y, z) = Q(x, y, z) = 2xy,$$

$$f_z(x, y, z) = x^2 + g_z(y, z) = R(x, y, z) = x^2 + 3z^2.$$

It follows that

$$g_y(y, z) = 0, \quad g_z(y, z) = 3z^2,$$

which yields

$$g(y, z) = z^3 + K$$

for some constant K. We conclude that $\mathbf{F} = \nabla f$ where

$$f(x, y, z) = x^2 z + y^2 x + z^3 + K$$

where K is an arbitrary constant.

To evaluate the line integral of $\mathbf{F}$ over the curve C parametrized by

$$x = t^2, \quad y = t + 1, \quad z = 2t - 1, \quad 0 \leq t \leq 1,$$

we apply the Fundamental Theorem of Line Integrals and obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= f(x(1), y(1), z(1)) - f(x(0), y(0), z(0)) \\
&= f(1, 2, 1) - f(0, 1, -1) \\
&= 1^2(1) + 2^2(1) + 1^3 + K - (0^2(-1) + 1^2(0) + (-1)^3 + K) \\
&= 1 + 4 + 1 + K - (0 + 0 - 1 + K) \\
&= 7.
\end{aligned}$$

□

Let $\mathbf{F}$ represent a force field. Then, recall that the work done by the force field to move an object along a path $\mathbf{r}(t)$, $a \leq t \leq b$, is given by the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

From Newton's Second Law of Motion, we have

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t),$$

where m is the mass of the object, and $\mathbf{r}''(t) = \mathbf{a}(t)$ is its acceleration. We then have

$$\begin{aligned}
W &= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt \\
&= \frac{1}{2} m \int_a^b \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt \\
&= \frac{1}{2} m \int_a^b \frac{d}{dt}[\|\mathbf{r}'(t)\|^2]\, dt \\
&= \frac{1}{2} m\|\mathbf{v}(b)\|^2 - \frac{1}{2} m\|\mathbf{v}(a)\|^2
\end{aligned}$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity of the object.

It follows that

$$W = K(B) - K(A),$$

where $A = \mathbf{r}(a)$ and $B = \mathbf{r}(b)$ are the initial and terminal points, respectively, and

$$K(P) = \frac{1}{2}m\|\mathbf{v}(t)\|, \quad \mathbf{r}(t) = P,$$

is the *kinetic energy* of the object at the point P. That is, the work done by the force field along C is the change in the kinetic energy from point A to point B.

If $\mathbf{F}$ is also a conservative force field, then $\mathbf{F} = -\nabla P$, where P is the *potential energy*. It follows from the Fundamental Theorem of Line Integrals that

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(B) - P(A)].$$

We conclude that

$$P(A) + K(A) = P(B) + K(B).$$

That is, when an object is moved by a conservative force field, then its total energy remains constant. This is known as the *Law of Conservation of Energy*.

3.4 Green's Theorem

We have learned that if a vector field is conservative, then its line integral over a closed curve C is equal to zero. However, if this is not the case, then evaluation of a line integral using the formula

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where $\mathbf{r}(t)$ is a parameterization of C, can be very difficult, even if C is a relatively simple plane curve. Fortunately, in this case, there is an alternative approach, using a result known as *Green's Theorem*.

We assume that $\mathbf{F} = \langle P, Q\rangle$, and consider the case where C encloses a region D that can be viewed as a region of *either* type I or type II. That is, D has the definitions

$$D = \{(x, y) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

and

$$D = \{(x,y)\,|\,c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$

Using the first definition, we have $C = C_1 \cup C_2 \cup (-C_3) \cup (-C_4)$, where:

- C_1 is the curve with parameterization $x = t$, $y = g_1(t)$, for $a \le t \le b$
- C_2 is the vertical line segment with parameterization $x = b$, $y = t$, for $g_1(b) \le t \le g_2(b)$
- C_3 is the curve with parameterization $x = t$, $y = g_2(t)$, for $a \le t \le b$
- C_4 is the vertical line segment with parameterization $x = a$, $y = t$, for $g_1(a) \le t \le g_2(a)$

We use *positive orientation* to describe the curve C, which means that the curve is traversed *counterclockwise*. This means that as the curve is traversed, the region D is "on the left".

In view of

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy,$$

we have

$$\begin{aligned}
\int_C P\,dx &= \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{-C_3} P\,dx + \int_{-C_4} P\,dx \\
&= \int_{C_1} P\,dx + \int_{C_2} P\,dx - \int_{C_3} P\,dx - \int_{C_4} P\,dx \\
&= \int_a^b P(x(t),y(t))x'(t)\,dt + \int_{g_1(b)}^{g_2(b)} P(x(t),y(t))x'(t)\,dt - \\
&\quad \int_a^b P(x(t),y(t))x'(t)\,dt - \int_{g_1(a)}^{g_2(a)} P(x(t),y(t))x'(t)\,dt \\
&= \int_a^b P(t,g_1(t))(1)\,dt + \int_{g_1(b)}^{g_2(b)} P(b,t)(0)\,dt - \\
&\quad \int_a^b P(t,g_2(t))(1)\,dt - \int_{g_1(a)}^{g_2(a)} P(a,t)(0)\,dt \\
&= \int_a^b [P(t,g_1(t)) - P(t,g_2(t))]\,dt \\
&= -\int_a^b \int_{g_1(t)}^{g_2(t)} P_y(t,y)\,dy\,dt \\
&= -\int\int_D \frac{\partial P}{\partial y}\,dA.
\end{aligned}$$

Using a similar approach in which D is viewed as a region of type II, we obtain

$$\int_C Q\,dy = \int\int_D \frac{\partial Q}{\partial x}\,dA.$$

Putting these results together, we obtain *Green's Theorem*, which states that if C is a positively oriented, piecewise smooth, simple (that is, not self-intersecting) closed curve that encloses a region D, and P and Q are functions that have continuous first partial derivatives on D, then

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Another common statement of the theorem is

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \int_{\partial D} P\,dx + Q\,dy,$$

where ∂D denotes the positively oriented boundary of D.

This theorem can be used to find a simpler approach to evaluating a line integral of the vector field $\langle P, Q\rangle$ over C by converting the integral to a double integral over D, or it can be used to find a simpler approach to evaluating a double integral over a region D by converting it into an integral over its boundary.

To show that Green's Theorem applies for more general regions than those that are of both type I and type II, we consider a region D that is the union of two regions D_1 and D_2 that are of both type I and type II. Let C be the positively oriented boundary of D, let D_1 have positively oriented boundary $C_1 \cup C_3$, and let D_2 have positively oriented boundary $C_2 \cup (-C_3)$, where C_3 is the boundary between D_1 and D_2. Then, $C = C_1 \cup C_2$. It follows that for functions P and Q that satisfy the assumptions of Green's Theorem on D, we can apply the theorem to D_1 and D_2 individually to obtain

$$\begin{aligned}
\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA &= \int\int_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA + \\
&\quad \int\int_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA \\
&= \int_{C_1 \cup C_3} P\,dx + Q\,dy + \int_{C_2 \cup (-C_3)} P\,dx + Q\,dy \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \\
&\quad \int_{C_2} P\,dx + Q\,dy + \int_{-C_3} P\,dx + Q\,dy
\end{aligned}$$

$$\begin{aligned}
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \\
&\quad \int_{C_3} P\,dx + Q\,dy - \int_{C_3} P\,dx + Q\,dy \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&= \int_{C_1 \cup C_2} P\,dx + Q\,dy \\
&= \int_{C} P\,dx + Q\,dy.
\end{aligned}$$

We conclude that Green's Theorem holds on $D_1 \cup D_2$. The same argument can be used to easily show that Green's Theorem applies on any finite union of *simple regions*, which are regions of both type I and type II.

Green's Theorem can also be applied to regions with "holes", that is, regions that are not simply connected. To see this, let D be a region enclosed by two curves C_1 and C_2 that are both positively oriented with respect to D (that is, D is on the left as either C_1 or C_2 is traversed). Let C_2 be contained within the region enclosed by C_1; that is, let C_2 be the boundary of the "hole" in D. Then, we can decompose D into two simply connected regions D' and D'' by connecting C_2 to C_1 along two separate curves that lie within D. Applying Green's Theorem to D' and D'' individually, we find that the line integrals along the common boundaries of D' and D'' cancel, because they have opposite orientations with respect to these regions. Therefore, we have

$$\begin{aligned}
\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \int\int_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \\
&\quad \int\int_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&= \int_{C_1 \cup C_2} P\,dx + Q\,dy.
\end{aligned}$$

Therefore, Green's Theorem applies to D as well.

Example The vector field

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

is conservative on all of $\mathbb{R}^2$ except at the origin, because it is not defined there. Specifically, $\mathbf{F} = \nabla f$ where

$$f(x, y) = \tan^{-1} \frac{y}{x}.$$

Now, consider a region D that is enclosed by a positively oriented, piecewise smooth, simple closed curve C, and also has a "hole" that is a disk of radius a, centered at the origin, and contained entirely within C. Let C' be the positively oriented boundary of this disk. Then, the boundary of D is $C \cup (-C')$, because, as a portion of the boundary of D, rather than the disk, it is necessary for C' to switch orientation. Applying Green's Theorem to compute the line integral of $\mathbf{F}$ over the boundary of D yields

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0,$$

since $\mathbf{F}$ is conservative on D. It follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

so we can compute the line integral of $\mathbf{F}$ over C, which we have not specified, by computing the line integral over the circle C', which can be parameterized by $x = a\cos t$, $y = a\sin t$, for $0 \leq t \leq 2\pi$. This yields

$$\begin{aligned}
\int_{C'} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} P(x(t), y(t))x'(t)\,dt + Q(x(t), y(t))y'(t)\,dt \\
&= \int_0^{2\pi} \left(-\frac{a\sin t}{(a\cos t)^2 + (a\sin t)^2} \right) (-a\sin t)\,dt + \\
&\quad \left(\frac{a\cos t}{(a\cos t)^2 + (a\sin t)^2} \right) (a\cos t)\,dt \\
&= \int_0^{2\pi} \frac{a^2\sin^2 t}{a^2\cos^2 + a^2\sin^2 t}\,dt + \frac{a^2\cos^2 t}{a^2\cos^2 t + a^2\sin^2 t}\,dt \\
&= \int_0^{2\pi} 1\,dt \\
&= 2\pi.
\end{aligned}$$

We conclude that the line integral of $\mathbf{F}$ over *any* positively oriented, piecewise smooth, simple closed curve that encloses the origin is equal to 2π. □

Example Consider a n-sided polygon P with vertices (x_1, y_1), (x_2, y_2), …, (x_n, y_n). The area A of the polygon is given by the double integral

$$A = \int\int_P 1\,dA.$$

Let $P(x,y) = -y/2$ and $Q(x,y) = x/2$. Then

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = 1.$$

It follows from Green's Theorem that if ∂P is positively oriented, then

$$A = \int_{\partial P} Q\,dy + P\,dx = \frac{1}{2}\int_{\partial P} x\,dy - y\,dx.$$

To evaluate this line integral, we consider each edge of P individually. Let C be the line segment from (x_1, y_1) to (x_2, y_2), and assume, for convenience, that C is not vertical. Then C can be parameterized by $x = t$, $y = mx + b$, for $x_1 \leq x \leq x_2$, where

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad b = y_1 - mx_1.$$

We then have

$$\begin{aligned}
\int_C x\,dy - y\,dx &= \int_{x_1}^{x_2} mt\,dt - (mt + b)\,dt \\
&= -\int_{x_1}^{x_2} b\,dt \\
&= b(x_1 - x_2) \\
&= y_1(x_1 - x_2) - mx_1(x_1 - x_2) \\
&= y_1(x_1 - x_2) + (y_2 - y_1)x_1 \\
&= x_1 y_2 - x_2 y_1.
\end{aligned}$$

We conclude that

$$\begin{aligned}
A &= \frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + \right. \\
&\quad \left. (x_ny_1 - x_1y_n)\right].
\end{aligned}$$

□

3.5 Curl and Divergence

We have seen two theorems in vector calculus, the Fundamental Theorem of Line Integrals and Green's Theorem, that relate the integral of a set to an integral over its boundary. Before establishing similar results that apply to surfaces and solids, it is helpful to introduce new operations on vector fields that will simplify exposition.

We have previously learned that a vector field $\mathbf{F} = \langle P, Q, R \rangle$ defined on $\mathbb{R}^3$ is conservative if

$$R_y - Q_z = 0, \quad P_z - R_x = 0, \quad Q_x - P_y = 0.$$

These equations are equivalent to the statement

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle = \langle 0, 0, 0 \rangle.$$

Therefore, we define the *curl* of a vector field $\mathbf{F} = \langle P, Q, R \rangle$ by

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F},$$

where

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

From the definition of a conservative vector field, it follows that $\operatorname{curl} \mathbf{F} = \mathbf{0}$ if $\mathbf{F} = \nabla f$ where f has continuous second partial derivatives, due to Clairaut's Theorem. That is, *the curl of a gradient is zero.*

This is equivalent to the statement that the curl of a conservative vector field is zero. The converse, that a vector field $\mathbf{F}$ for which $\operatorname{curl} \mathbf{F} = \mathbf{0}$ is conservative, is also true if $\mathbf{F}$ has continuous first partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$ within a *simply connected* domain. That is, the domain must not have "holes".

When $\mathbf{F}$ represents the velocity field of a fluid, the fluid tends to rotate around the axis that is aligned with $\operatorname{curl} \mathbf{F}$, and the magnitude of $\operatorname{curl} \mathbf{F}$ indicates the speed of rotation. Therefore, when $\operatorname{curl} \mathbf{F} = \mathbf{0}$, we say that $\mathbf{F}$ is *irrotational*, which is a term that has previously been associated with the equivalent condition of $\mathbf{F}$ being conservative.

Another operation that is useful for discussing properties of vector fields is the *divergence* of a vector field $\mathbf{F}$, denoted by $\operatorname{div} \mathbf{F}$. It is defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}.$$

For example, if $\mathbf{F} = \langle P, Q, R \rangle$, then

$$\operatorname{div} \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z.$$

Unlike the curl, the divergence is defined for vector fields with any number of variables, as long as the number of independent and the number of dependent variables are the same.

It can be verified directly that if $\mathbf{F}$ is the curl of a vector field $\mathbf{G}$, then $\operatorname{div}\mathbf{F} = 0$. That is, *the divergence of any curl is zero*, as long as $\mathbf{G}$ has continuous second partial derivatives. This is useful for determining whether a given vector field $\mathbf{F}$ is the curl of any other vector field $\mathbf{G}$, for if it is, its divergence must be zero.

Example (Stewart, Section 13.5, Exercise 18) The vector field $\mathbf{F}(x, y, z) = \langle yz, xyz, xy\rangle$ is not the curl of any vector field $\mathbf{G}$, because

$$\operatorname{div}\mathbf{F} = (yz)_x + (xyz)_y + (xy)_z = 0 + xz + 0 = xz,$$

whereas if $\mathbf{F} = \operatorname{curl}\mathbf{G}$, then

$$\operatorname{div}\mathbf{F} = \operatorname{div}\operatorname{curl}\mathbf{G} = 0.$$

□

If $\mathbf{F}$ represents the velocity field of a fluid, then, at each point within the fluid, $\operatorname{div}\mathbf{F}$ measures the tendency of the fluid to diverge away from that point. Specifically, the divergence is the rate of change, with respect to time, of the density of the fluid. Therefore, if $\operatorname{div}\mathbf{F} = 0$, then we say that $\mathbf{F}$, and therefore the fluid as well, is *incompressible*.

The divergence of a gradient is

$$\operatorname{div}(\nabla f) = \nabla\cdot\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

We denote this expression $\nabla\cdot\nabla f$ by $\nabla^2 f$, or Δf, which is called the *Laplacian* of f. The operator ∇^2 is called the *Laplace operator*. Its name comes from *Laplace's equation*

$$\Delta f = 0.$$

The curl and divergence can be used to restate Green's Theorem in forms that are more directly generalizable to surfaces and solids in $\mathbb{R}^3$. Let $\mathbf{F} = \langle P, Q, 0\rangle$, the embedding of a two-dimensional vector field in $\mathbb{R}^3$. Then

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k},$$

where, as before, $\mathbf{k} = \langle 0, 0, 1\rangle$. It follows that

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\cdot\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

This expression is called the *scalar curl* of the two-dimensional vector field $\langle P, Q\rangle$. We conclude that Green's Theorem can be rewritten as

$$\int_C \mathbf{F}\,d\mathbf{r} = \int\int_D (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\,dA.$$

Another useful form of Green's Theorem involves the divergence. Let $\mathbf{F} = \langle P, Q\rangle$ have continuous first partial derivatives in a domain D with a positively oriented, piecewise smooth boundary C that has parametrization $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, for $a \leq t \leq b$. Using the original form of Green's Theorem, we have

$$\begin{aligned}
\int\int_D \operatorname{div}\mathbf{F}\,dA &= \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \\
&= \int_C P\,dy - Q\,dx \\
&= \int_a^b P(x(t), y(t))y'(t)\,dt - Q(x(t), y(t))x'(t)\,dt \\
&= \int_a^b \left[P(x(t), y(t))\frac{y'(t)}{\|\mathbf{r}'(t)\|} + Q(x(t), y(t))\frac{-x'(t)}{\|\mathbf{r}'(t)\|}\right]\|\mathbf{r}'(t)\|\,dt \\
&= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\|\mathbf{r}'(t)\|\,dt \\
&= \int_C \mathbf{F}\cdot\mathbf{n}\,ds
\end{aligned}$$

where

$$\mathbf{n}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\langle y'(t), -x'(t)\rangle$$

is the *outward unit normal vector* to the curve C. Note that $\mathbf{n}\cdot\mathbf{T} = 0$, where $\mathbf{T}$ is the unit tangent vector

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\langle x'(t), y'(t)\rangle.$$

We have established a third form of Green's Theorem,

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int\int_D \operatorname{div}\mathbf{F}\,dA.$$

3.6 Parametric Surfaces and Their Areas

We have learned that Green's Theorem can be used to relate a line integral of a two-dimensional vector field $\mathbf{F}$ over a closed plane curve C to a double integral of a component of curl $\mathbf{F}$ over the region D that is enclosed by C. Our goal is to generalize this result in such a way as to relate the line integral of a *three-dimensional* vector field $\mathbf{F}$ over a closed *space curve* C to the integral of a component of curl $\mathbf{F}$ over a *surface* enclosed by C.

We have also learned that Green's Theorem relates the integral of the *normal component* of a two-dimensional vector field over a closed curve C to the double integral of div $\mathbf{F}$ over the region D that it encloses. We wish to generalize this result in order to relate the integral of the normal component of a three-dimensional vector field $\mathbf{F}$ over a *closed surface* S to the *triple integral* of div $\mathbf{F}$ over the solid E contained within S.

In order to realize either of these generalizations, we need to be able to integrate functions over piecewise smooth surfaces, just as we now know how to integrate functions over piecewise smooth curves. Whereas a smooth curve C, being a curved one-dimensional entity, is most conveniently described by a parameterization $\mathbf{r}(t)$, where $a \leq t \leq b$ and $\mathbf{r}(t)$ is a differentiable function of one variable, a smooth surface S, being a curved two-dimensional entity, is most conveniently described by a parametrization $\mathbf{r}(u,v)$, where (u,v) lies within a 2-D region, and $\mathbf{r}(u,v) = \langle x(u,v), y(y,v), z(u,v)\rangle$ is a differentiable function of two variables. We say that S is a parametric surface, and

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

are the *parametric equations* of S.

Example The *Möbius strip* is a surface that is famous for being a *nonorientable* surface; that is, it "has only one side". It can be parameterized by

$$\begin{aligned} x(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u, \\ y(u,v) &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u, \\ z(u,v) &= \frac{v}{2}\sin\frac{u}{2}, \end{aligned}$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$. It is shown in Figure 3.3. □

Example The paraboloid defined by the equation $x = 4y^2 + 4z^2$, $0 \leq x \leq 4$,

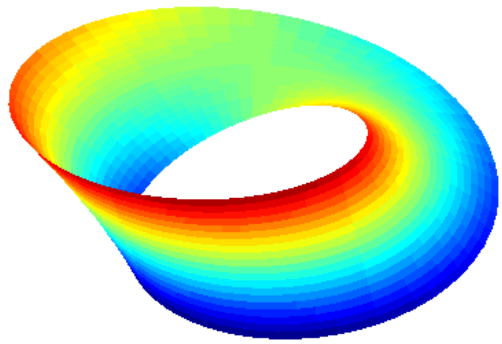

Figure 3.3: The Möbius strip

can also be defined by the parametric equations

$$x = x, \quad y = \frac{\sqrt{x}}{2}\cos\theta, \quad z = \frac{\sqrt{x}}{2}\sin\theta,$$

where $0 \le \theta \le 2\pi$, since for each x, a point (x, y, z) on the paraboloid must lie on a circle centered at $(x, 0, 0)$ with radius $\sqrt{x/4}$, parallel to the yz-plane. This is an example of a *surface of revolution*, since the surface is obtained by revolving the curve $y = f(x)$ around the x-axis. □

Let $P_0 = (x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ be a point on a parametric surface S. A curve defined by $\mathbf{g}(v) = \mathbf{r}(u_0, v)$ that lies within S and passes through P_0 has the tangent vector

$$\mathbf{r}_v = \mathbf{g}'(v) = \left\langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle$$

at P_0. Similarly, the tangent vector at P_0 of the curve $\mathbf{h}(u) = \mathbf{r}(u, v_0)$, that also lies within S and passes through P_0, is

$$\mathbf{r}_u = \mathbf{h}'(u) = \left\langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle.$$

If these vectors are not parallel, then together they define the *tangent plane* of S at P_0. Its normal vector is

$$\mathbf{r}_u \times \mathbf{r}_v = \langle a, b, c \rangle$$

which yields the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

of the tangent plane.

Example (Stewart, Section 13.6, Exercise 30) Consider the surface defined by the parametric equations

$$x = u^2, \quad y = v^2, \quad z = uv, \quad 0 \leq u, v \leq 10.$$

At the point $(x_0, y_0, z_0) = (1, 1, 1)$, which corresponds to $u_0 = 1$, $v_0 = 1$, the equation of the tangent plane can be obtained by first computing the partial derivatives of the coordinate functions. We have

$$x_u = 2u, \quad y_u = 0, \quad z_u = v,$$

$$x_v = 0, \quad y_v = 2v, \quad z_v = u.$$

Evaluating at (u_0, v_0) yields

$$\mathbf{r}_u = \langle x_u, y_u, z_u \rangle = \langle 2, 0, 1 \rangle, \quad \mathbf{r}_v = \langle x_v, y_v, z_v \rangle = \langle 0, 2, 1 \rangle.$$

It follows that the normal to the tangent plane is

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \langle 2, 0, 1 \rangle \times \langle 0, 2, 1 \rangle = \langle -2, -2, 4 \rangle.$$

We conclude that the equation of the tangent plane is

$$-2(x - 1) - 2(y - 1) + 4(z - 1) = 0.$$

□

The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are helpful for computing the area of a smooth surface S. For simplicity, we assume that S is parametrized by a function $\mathbf{r}(u, v)$ with domain D, where $D = [a, b] \times [c, d]$ is a rectangle in the uv-plane. We divide $[a, b]$ into n subintervals $[u_{i-1}, u_i]$ of width $\Delta u = (b - a)/n$, and divide $[c, d]$ into m subintervals $[v_{j-1}, v_j]$ of width $\Delta v = (d - c)/m$.

Then, $\mathbf{r}$ approximately maps the rectangle R_{ij} with lower left corner (u_{i-1}, v_{j-1}) into a parallelogram with adjacent edges defined by the vectors

$$\mathbf{r}(u_i, v_{j-1}) - \mathbf{r}(u_{i-1}, v_{j-1}) \approx \mathbf{r}_u \Delta u$$

and

$$\mathbf{r}(u_{i-1}, v_j) - \mathbf{r}(u_{i-1}, v_{j-1}) \approx \mathbf{r}_v \Delta v.$$

The area of this parallelogram is

$$A_{ij} = \|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v.$$

Adding all of these areas approximates the area of S, which we denote by $A(S)$. If we let $m, n \to \infty$, we obtain

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij} = \int\int_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA.$$

Example (Stewart, Section 13.6, Exercise 34) We wish to find the area of the surface S that is the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. First, we must find parametric equations for this surface. Because x and y are restricted to the circle of radius 3 centered at the origin, it makes sense to use polar coordinates for x and y. We then have the parametric equations

$$x = u \cos v, \quad y = u \sin v, \quad z = 10 - u(2 \cos v + 5 \sin v),$$

where $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. We then have

$$\mathbf{r}_u = \langle x_u, y_u, z_u \rangle = \langle \cos v, \sin v, -2 \cos v - 5 \sin v \rangle,$$

$$\mathbf{r}_v = \langle x_v, y_v, z_v \rangle = \langle -u \sin v, u \cos v, u(2 \sin v - 5 \cos v) \rangle.$$

We then have

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|\langle 2u, 5u, u \rangle\| = |u|\sqrt{30}.$$

It follows that

$$A(S) = \int_0^3 \int_0^{2\pi} u\sqrt{30} \, du \, dv = 2\pi\sqrt{30} \int_0^3 u \, du = 9\pi\sqrt{30}.$$

It should be noted that it is to be expected that the direction of $\mathbf{r}_u \times \mathbf{r}_v$ is parallel to the normal vector of the plane $2x+5y+z = 10$, since it is normal to the surface at every point. □

Often, a surface is defined to be the graph of a function $z = f(x, y)$. Such a surface can be parametrized by

$$x = u, \quad y = v, \quad z = f(u, v), \quad (u, v) \in D.$$

It follows that

$$\mathbf{r}_u = \langle 1, 0, f_u \rangle, \quad \mathbf{r}_v = \langle 0, 1, f_v \rangle.$$

We then have $\mathbf{r}_v \times \mathbf{r}_u = \langle f_u, f_v, -1 \rangle$, which yields the equation of the tangent plane

$$\frac{\partial f}{\partial u}(u_0, v_0)(x - x_0) + \frac{\partial f}{\partial v}(u_0, v_0)(y - y_0) = z - z_0,$$

which, using the relations $x = u$ and $y = v$, can be rewritten as

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = z - z_0.$$

Recall that this is the equation of the tangent plane of a surface defined by an equation of the form $z = f(x, y)$ that had been previously defined. It follows that the area of such a surface is given by the double integral

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}\, dA.$$

Example (Stewart, Section 13.6, Exercise 38) To find the area $A(S)$ of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$ and $(2, 1)$, we compute

$$\frac{\partial z}{\partial x} = 3, \quad \frac{\partial z}{\partial y} = 4y,$$

and then evaluate the double integral

$$\begin{aligned} A(S) &= \int_0^1 \int_0^{2y} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \\ &= \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy \\ &= \int_0^1 2y\sqrt{10 + 16y^2}\, dy \\ &= \frac{1}{16} \int_{10}^{26} u^{1/2}\, du \end{aligned}$$

$$\begin{aligned} &= \frac{1}{24}u^{3/2}\Big|_{10}^{26} \\ &= \frac{1}{24}(26^{3/2} - 10^{3/2}) \\ &\approx 4.206. \end{aligned}$$

□

A surface of revolution S that is obtained by revolving the curve $y = f(x)$, $a \leq x \leq b$, around the x-axis has parametric equations

$$x = u, \quad y = f(u)\cos v, \quad z = f(u)\sin v,$$

where $a \leq u \leq b$ and $0 \leq v \leq 2\pi$. From these equations, we obtain

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = |f(u)|\sqrt{1 + [f'(u)]^2},$$

which yields

$$A(S) = 2\pi \int_a^b |f(u)|\sqrt{1 + [f'(u)]^2}\,du.$$

If $y = f(x)$ is revolved around the y-axis instead, then the area is

$$A(S) = 2\pi \int_a^b |u|\sqrt{1 + [f'(u)]^2}\,du,$$

which can be obtained by considering the case of revolving $x = f^{-1}(y)$ around the y-axis and proceeding with a parametrization similar to the case of revolving around the x-axis.

3.7 Surface Integrals

3.7.1 Surface Integrals of Scalar-Valued Functions

Previously, we have learned how to integrate functions along curves. If a smooth space curve C is parameterized by a function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$, then the arc length L of C is given by the integral

$$\int_a^b \|\mathbf{r}'(t)\|\,dt.$$

Similarly, the integral of a scalar-valued function $f(x, y, z)$ along C is given by

$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|\,dt.$$

It follows that the integral of $f(x, y, z) \equiv 1$ along C is equal to the arc length of C.

We now define integrals of scalar-valued functions on surfaces in an analogous manner. Recall that the area of a smooth surface S, parametrized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ for $(u, v) \in D$, is given by the integral

$$A(S) = \int\int_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv.$$

To integrate a scalar-valued function $f(x, y, z)$ over S, we assume for simplicity that D is a rectangle, and divide it into sub-rectangles $\{R_{ij}\}$ of dimension Δu and Δv, as we did when we derived the formula for $A(S)$. Then, the function $\mathbf{r}$ maps each sub-rectangle R_{ij} into a surface patch S_{ij} that has area ΔS_{ij}. This area is then multiplied by $f(P^*_{ij})$, where P^*_{ij} is any point on S_{ij}.

Letting $\Delta u, \Delta v \to 0$, we obtain the *surface integral* of f over S to be

$$\begin{aligned} \int\int_S f(x, y, z) \, dS &= \lim_{\Delta u, \Delta v \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(P^*_{ij}) \, \Delta S_{ij} \\ &= \int\int_D f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv, \end{aligned}$$

since, in the limit as $\Delta u, \Delta v \to 0$, we have

$$\Delta S_{ij} \to \|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta u \, \Delta v.$$

Note that if $f(x, y, z) \equiv 1$, then the surface integral of f over S yields the area of S, $A(S)$.

Example (Stewart, Section 13.7, Exercise 6) Let S be the helicoid with parameterization

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v\rangle, \quad 0 \le u \le 1, \quad 0 \le v \le \pi.$$

Then we have

$$\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1\rangle,$$

which yields

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|\langle \sin v, -\cos v, u\rangle\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}.$$

It follows that

$$\begin{aligned}
\int\int_S \sqrt{1+x^2+y^2}\,dS &= \int_0^1\int_0^\pi \sqrt{1+(u\cos v)^2+(u\sin v)^2}\|\mathbf{r}_u\times\mathbf{r}_v\|\,dv\,du \\
&= \int_0^1\int_0^\pi \sqrt{1+u^2}\sqrt{1+u^2}\,dv\,du \\
&= \int_0^1\int_0^\pi 1+u^2\,dv\,du \\
&= \pi\int_0^1 1+u^2\,du \\
&= \pi\left(u+\frac{u^3}{3}\right)\Big|_0^1 \\
&= \frac{4\pi}{3}.
\end{aligned}$$

□

The surface integral of a scalar-valued function is useful for computing the mass and center of mass of a thin sheet. If the sheet is shaped like a surface S, and it has density $\rho(x,y,z)$, then the mass is given by the surface integral

$$m = \int\int_S \rho(x,y,z)\,dS,$$

and the center of mass is the point $(\bar{x},\bar{y},\bar{z})$, where

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int\int_S x\rho(x,y,z)\,dS, \\
\bar{y} &= \frac{1}{m}\int\int_S y\rho(x,y,z)\,dS, \\
\bar{z} &= \frac{1}{m}\int\int_S z\rho(x,y,z)\,dS.
\end{aligned}$$

3.7.2 Surface Integrals of Vector Fields

Let $\mathbf{v}$ be a vector field defined on $\mathbb{R}^3$ that represents the velocity field of a fluid, and let ρ be the density of the fluid. Then, the rate of *flow* of the fluid, which is defined to be the rate of change with respect to time of the amount of fluid (mass), per unit area, is given by $\rho\mathbf{v}$.

To determine the total amount of fluid that is crossing S per unit of time, called the *flux* across S, we divide S into several small patches S_{ij}, as we did when we defined the surface integral of a scalar-valued function. Since each patch S_{ij} is approximately planar (that is, parallel to a plane), we can approximate the flux across S_{ij} by

$$(\rho\mathbf{v}\cdot\mathbf{n})A(S_{ij}),$$

where $\mathbf{n}$ is a unit vector that is normal (perpendicular) to S_{ij}. This is because if θ is the angle between S_{ij} and the direction of $\mathbf{v}$, then the fluid directed at S_{ij} is effectively passing through a region of area $A(S_{ij})|\cos\theta|$.

If we sum the flux over each patch, and let the areas of the patches approach zero, then we obtain the total flux across S,

$$\int\int_S \rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)\,dS,$$

where $\mathbf{n}(x,y,z)$ is a continuous function that describes a unit normal vector at each point (x,y,z) on S. For a general vector field $\mathbf{F}$, we define the *surface integral* of $\mathbf{F}$ over S by

$$\int\int_S \mathbf{F}\cdot\,d\mathbf{S} = \int\int_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

When $\mathbf{F}$ represents an electric field, we call the surface integral of $\mathbf{F}$ over S the *electric flux* of $\mathbf{F}$ through S. Alternatively, if $\mathbf{F} = -K\nabla u$, where u is a function that represents temperature and K is a constant that represents thermal conductivity, then the surface integral of $\mathbf{F}$ over a surface S is called the *heat flow* or *heat flux* across S.

If S is parameterized by a function $\mathbf{r}(u,v)$, where $(u,v)\in D$, then

$$\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|},$$

and we then have

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\,d\mathbf{S} &= \int\int_S \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\,dS \\ &= \int\int_D \mathbf{F}(\mathbf{r}(u,v))\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA \\ &= \int\int_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.\end{aligned}$$

This is analogous to the definition of the line integral of a vector field over a curve C,

$$\int_C \mathbf{F}\cdot\,d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.$$

Just as the orientation of a curve was relevant to the line integral of a vector field over a curve, the orientation of a surface is relevant to the surface integral of a vector field. We say that a surface S is *orientable*, or *oriented*, if, at each point (x, y, z) in S, it is possible to choose a *unique* vector $\mathbf{n}(x, y, z)$ that is normal to the tangent plane of S at (x, y, z), in such a way that $\mathbf{n}(x, y, z)$ varies *continuously* over S. The particular choice of $\mathbf{n}$ is called an *orientation*.

An orientable surface has two orientations, or, informally, two "sides", with normal vectors $\mathbf{n}$ and $-\mathbf{n}$. This definition of orientability excludes the Möbius strip, because for this surface, it is possible for a continuous variation of (x, y, z) to yield two distinct normal vectors at every point of the surface, that are negatives of one another. Geometrically, the Möbius strip can be said to have only one "side", because negating any choice of continuously varying $\mathbf{n}$ yields the same normal vectors.

For a surface that is the graph of a function $z = g(x, y)$, if we choose the parametrization

$$x = u, \quad y = v, \quad z = g(u, v),$$

then from

$$\mathbf{r}_u = \langle 1, 0, g_u \rangle, \quad \mathbf{r}_v = \langle 0, 1, g_v \rangle,$$

we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -g_u, -g_v, 1 \rangle = \langle -g_x, -g_y, 1 \rangle$$

which yields

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + g_x^2 + g_y^2}}.$$

Because the z-component of this vector is positive, we call this choice of $\mathbf{n}$ an *upward* orientation of the surface, while $-\mathbf{n}$ is a *downward* orientation.

Example (Stewart, Section 13.7, Exercise 22) Let S be the part of the cone $z = \sqrt{x^2 + y^2}$ that lies beneath the plane $z = 1$, with *downward* orientation. We wish to evaluate the surface integral

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = \langle x, y, z^4 \rangle$.

First, we must compute the unit normal vector for S. Using cylindrical coordinates yields the parameterization

$$x = u \cos v, \quad y = u \sin v, \quad z = u, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

We then have

$$\mathbf{r}_u = \langle \cos v, \sin v, 1\rangle, \quad \mathbf{r}_v = \langle -u\sin v, u\cos v, 0\rangle,$$

which yields

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -u\cos v, -u\sin v, u\cos^2 v + u\sin^2 v\rangle = u\langle -\cos v, -\sin v, 1\rangle.$$

Because we assume downward orientation, we must have the z-component of the normal vector be negative. Therefore, $\mathbf{r}_u \times \mathbf{r}_v$ must be negated, which yields

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = -\int\int_D \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA,$$

where D is the domain of the parameters u and v, the rectangle $[0,1]\times[0,2\pi]$. Evaluating this integral, we obtain

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= -\int\int_D \langle u\cos v, u\sin v, u^4\rangle \cdot u\langle -\cos v, -\sin v, 1\rangle\, dA \\
&= -\int_0^{2\pi}\int_0^1 (-u\cos^2 v - u\sin^2 v + u^4)u\, du\, dv \\
&= \int_0^{2\pi}\int_0^1 (u^2 - u^5)\, du\, dv \\
&= 2\pi \int_0^1 (u^2 - u^5)\, du \\
&= 2\pi \left(\frac{u^3}{3} - \frac{u^6}{6}\right)\Big|_0^1 \\
&= \frac{\pi}{3}.
\end{aligned}$$

An alternative approach is to retain Cartesian coordinates, and then use the formula for the unit normal for a downward orientation of a surface that is the graph of a function $z = g(x,y)$,

$$\mathbf{n} = -\frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}} = \frac{1}{\sqrt{2}}\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle.$$

This approach stil requires a conversion to polar coordinates to integrate over the unit disk in the xy-plane. □

For a *closed* surface S, which is the boundary of a solid region E, we define the *positive orientaion* of S to be the choice of $\mathbf{n}$ that consistently point *outward* from E, while the inward-pointing normals define the negative orientation.

Example (Stewart, Section 13.7, Exercise 26) To evaluate the surface integral

$$\int\int_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}(x,y,z)=\langle y, z-y, x\rangle$ and S is the surface of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, we must evaluate surface integrals over each of the four faces of the tetrahedron separately. We assume positive (outward) orientation.

For the first side, S_1, with vertices $(0,0,0)$, $(1,0,0)$ and $(0,0,1)$, we first parameterize the side using

$$x=u,\quad y=0,\quad z=v,\quad 0\leq u\leq 1,\quad 0\leq v\leq 1-u.$$

Then, from

$$\mathbf{r}_u=\langle 1,0,0\rangle,\quad \mathbf{r}_v=\langle 0,0,1\rangle,$$

we obtain

$$\mathbf{r}_u\times\mathbf{r}_v=\langle 0,-1,0\rangle.$$

This vector is pointing outside the tetrahedron, so it is the outward normal vector that we wish to use. Therefore, the surface integral of $\mathbf{F}$ over S_1 is

$$\begin{aligned}
\int\int_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^{1-u}\langle 0, v-0, u\rangle\cdot\langle 0,-1,0\rangle\,dv\,du \\
&= -\int_0^1\int_0^{1-u} v\,dv\,du \\
&= -\int_0^1 \left.\frac{v^2}{2}\right|_0^{1-u} du \\
&= -\frac{1}{2}\int_0^1 (1-u)^2\,du \\
&= \frac{1}{2}\left.\frac{(1-u)^3}{3}\right|_0^1 \\
&= -\frac{1}{6}.
\end{aligned}$$

For the second side, S_2, with vertices $(0,0,0)$, $(0,1,0)$ and $(0,0,1)$, we parameterize using

$$x = 0, \quad y = u, \quad z = v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1-u.$$

Then, from

$$\mathbf{r}_u = \langle 0,1,0\rangle, \quad \mathbf{r}_v = \langle 0,0,1\rangle,$$

we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1,0,0\rangle.$$

This vector is pointing *inside* the tetrahedron, so we must negate it to obtain the outward normal vector. Therefore, the surface integral of $\mathbf{F}$ over S_2 is

$$\begin{aligned}
\int\int_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{1-u} \langle u, v-u, 0\rangle \cdot \langle -1,0,0\rangle \, dv\, du \\
&= -\int_0^1 \int_0^{1-u} u\, dv\, du \\
&= \int_0^1 u(u-1)\, du \\
&= \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Big|_0^1 \\
&= -\frac{1}{6}.
\end{aligned}$$

For the base S_3, with vertices $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$, we parametrize using

$$x = u, \quad y = v, \quad z = 0, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1-u.$$

Then, from

$$\mathbf{r}_u = \langle 1,0,0\rangle, \quad \mathbf{r}_v = \langle 0,1,0\rangle,$$

we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0,0,1\rangle.$$

This vector is pointing *inside* the tetrahedron, so we must negate it to obtain the outward normal vector. Therefore, the surface integral of $\mathbf{F}$ over S_3 is

$$\begin{aligned}
\int\int_{S_3} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{1-u} \langle v, 0-v, u\rangle \cdot \langle 0,0,-1\rangle \, dv\, du \\
&= -\int_0^1 \int_0^{1-u} u\, dv\, du
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 u(u-1)\,du \\
&= \left.\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\right|_0^1 \\
&= -\frac{1}{6}.
\end{aligned}
$$

Finally, for the "top" face S_4, with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, we parametrize using

$$x = u, \quad y = v, \quad z = 1-u-v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1-u,$$

since the equation of the plane containing this face is $x+y+z-1=0$. This can be determined by using the three vertices to obtain two vectors within the plane, and then computing their cross product to obtain the plane's normal vector.

Then, from

$$\mathbf{r}_u = \langle 1, 0, -1\rangle, \quad \mathbf{r}_v = \langle 0, 1, -1\rangle,$$

we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 1\rangle.$$

This vector is pointing outside the tetrahedron, so it is the outward normal vector that we wish to use. Therefore, the surface integral of $\mathbf{F}$ over S_4 is

$$
\begin{aligned}
\int\int_{S_4} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{1-u} \langle v, 1-u-2v, u\rangle \cdot \langle 1,1,1\rangle \,dv\,du \\
&= \int_0^1 \int_0^{1-u} 1 - v\,dv\,du \\
&= \int_0^1 \left.\left(v - \frac{v^2}{2}\right)\right|_0^{1-u} du \\
&= \int_0^1 1 - u - \frac{(1-u)^2}{2}\,du \\
&= \int_0^1 \frac{1}{2} - \frac{1}{2}u^2\,du \\
&= \left.\left(\frac{u}{2} - \frac{u^3}{6}\right)\right|_0^1 \\
&= \frac{1}{3}.
\end{aligned}
$$

Adding the four integrals together yields

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{3} = -\frac{1}{6}.$$

□

3.8 Stokes' Theorem

Let C be a simple, closed, positively oriented, piecewise smooth plane curve, and let D be the region that it encloses. According to one of the forms of Green's Theorem, for a vector field $\mathbf{F}$ with continuous first partial derivatives on D, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D (\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dA,$$

where $\mathbf{k} = \langle 0,0,1\rangle$.

By noting that $\mathbf{k}$ is normal to the region D when it is embedded in 3-D space, we can generalize this form of Green's Theorem to more general surfaces that are enclosed by simple, closed, piecewise smooth, positively oriented *space* curves. Let S be an oriented, piecewise smooth surface that is enclosed by a such a curve C. If we divide S into several small patches S_{ij}, then these patches are approximately planar. We can apply Green's Theorem, approximately, to each patch by rotating it in space so that its unit normal vector is $\mathbf{k}$, and using the fact that rotating two vectors $\mathbf{u}$ and $\mathbf{v}$ in space does not change the value of $\mathbf{u}\cdot\mathbf{v}$.

Most of the line integrals along the boundary curves of each path cancel with one another due to the positive orientation of all such boundary curves, and we are left with the line integral over C, the boundary of S. If we take the limit as the size of the patches approches zero, we then obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is the unit normal vector of S. This result is known as *Stokes' Theorem*.

Stokes' Theorem can be used either to evaluate an surface integral or an integral over the curve that encloses it, whichever is easier.

Example (Stewart, Section 13.8, Exercise 2) Let $\mathbf{F}(x,y,z) = \langle yz, xz, xy\rangle$ and let S be the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, with upward orientation. By Stokes' Theorem,

$$\int\int_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r}$$

where C is the boundary curve of S, which is a circle of radius 2 centered at $(0, 0, 5)$, and parallel to the xy-plane. It can therefore be parameterized by

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 5, \quad 0 \leq t \leq 2\pi.$$

Its tangent vector is then

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle.$$

We then have

$$\begin{aligned}
\int\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\
&= \int_0^{2\pi} \langle 10\sin t, 10\cos t, 4\cos t\sin t\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle\,dt \\
&= \int_0^{2\pi} -20\sin^2 t + 20\cos^2 t\,dt \\
&= 20\int_0^{2\pi} \cos 2t\,dt \\
&= 10\sin 2t|_0^{2\pi} \\
&= 0.
\end{aligned}$$

This result can also be obtained by noting that because $\mathbf{F} = \nabla f$, where $f(x, y, z) = xyz$, it follows that $\operatorname{curl}\mathbf{F} = \mathbf{0}$. □

Example (Stewart, Section 13.8, Exercise 8) We wish to evaluate the line integral of $\mathbf{F}(x, y, z) = \langle xy, 2z, 3y\rangle$ over the curve C that is the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $x + z = 5$.

To describe the surface S enclosed by C, we use the parameterization

$$x = u\cos v, \quad y = u\sin v, \quad z = 5 - u\cos v, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

Using

$$\mathbf{r}_u = \langle \cos v, \sin v, -\cos v\rangle, \quad \mathbf{r}_v = \langle -u\sin v, u\cos v, u\sin v\rangle,$$

we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \langle u, 0, u\rangle.$$

We then compute

$$\operatorname{curl}\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle xy, 2z, 3y\rangle = \langle 1, 0, -x\rangle.$$

Let D be the domain of the parameters,

$$D = \{(u,v) \,|\, 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

We then apply Stokes' Theorem and obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\
&= \int\int_D \operatorname{curl} \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \\
&= \int_0^3 \int_0^{2\pi} \langle 1, 0, -u\cos v\rangle \cdot \langle u, 0, u\rangle \, dA \\
&= \int_0^3 \int_0^{2\pi} u - u^2 \cos v \, dv \, du \\
&= \int_0^3 (uv - u^2 \sin v)\big|_0^{2\pi} \, dv \, du \\
&= 2\pi \int_0^3 u \, du \, du \\
&= 2\pi \left. \frac{u^2}{2} \right|_0^3 \\
&= 9\pi.
\end{aligned}$$

□

Stokes' Theorem can also be used to provide insight into the physical interpretation of the curl of a vector field. Let S_a be a disk of radius a centered at a point P_0, and let C_a be its boundary. Furthermore, let $\mathbf{v}$ be a velocity field for a fluid. Then the line integral

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \int_{C_a} \mathbf{v} \cdot \mathbf{T} \, ds,$$

where $\mathbf{T}$ is the unit tangent vector of C_a, measures the tendency of the fluid to move around C_a. This is because this measure, called the *circulation* of $\mathbf{v}$ around C_a, is greatest when the fluid velocity vector is consistently parallel to the unit tangent vector. That is, the circulation around C_a is maximized when the fluid follows the path of C_a.

Now, by Stokes' Theorem,

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \int\int_{S_a} \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}$$

$$\begin{aligned} &= \int\int_{S_a} \operatorname{curl} \mathbf{v} \cdot \mathbf{n} \, dS \\ &\approx \operatorname{curl} \mathbf{V}(P_0) \cdot \mathbf{n}(P_0) \int\int_{S_a} 1 \, dS \\ &\approx \pi a^2 \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0). \end{aligned}$$

As $a \to 0$, and S_a collapses to the point P_0, this approximation improves, and we obtain

$$\operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}.$$

This shows that circulation is maximized when the axis around which the fluid is circulating, $\mathbf{n}(P_0)$, is parallel to $\operatorname{curl} \mathbf{v}$. That is, the direction of $\operatorname{curl} \mathbf{v}$ indicates the axis around which the greatest circulation occurs.

3.8.1 A Note About Orientation

Recall Stokes' Theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where C is a simple, closed, positively oriented, piecewise smooth curve and S is a oriented surface enclosed by C. If C is parameterized by a function $\mathbf{r}(t)$, where $a \leq t \leq b$, and S is parameterized by a function $\mathbf{g}(u, v)$, where $(u, v) \in D$, then Stokes' Theorem becomes

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int\int_D \operatorname{curl} \mathbf{F}(\mathbf{g}(u, v)) \cdot (\mathbf{g}_u \times \mathbf{g}_v) \, du \, dv.$$

It is important that the parameterizations $\mathbf{r}$ and $\mathbf{g}$ have the proper orientation for Stokes' Theorem to apply. This is why it is required that C have positive orientation. It means, informally, that if one were to "walk" along C, in such a way that $\mathbf{n}$, the unit normal vector of S, can be viewed, then S should always be "on the left" relative to the path traced along C.

It follows that the parameterizations of C and S must be consistent with one another, to ensure that they are oriented properly. Otherwise, one of the parameterizations must be reversed, so that the sign of the corresponding integral is corrected. The orientation of a curve can be reversed by changing the parameter to $s = a + b - t$. The orientation of a surface can be reversed by interchanging the variables u and v.

3.9 The Divergence Theorem

Let $\mathbf{F}$ be a vector field with continuous first partial derivatives. Recall a statement of Green's Theorem,

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int\int_D \operatorname{div}\mathbf{F}\,dA,$$

where $\mathbf{n}$ is the outward unit normal vector of D. Now, let E be a three-dimensional solid whose boundary, denoted by ∂E, is a closed surface S with positive orientation. Then, if we consider two-dimensional slices of E, each one being parallel to the xy-plane, then each slice is a region D with positively oriented boundary C, to which Green's Theorem applies. If we multiply the integrals on both sides of Green's Theorem, as applied to each slice, by dz, the infinitesimal "thickness" of each slice, then we obtain

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int\int_E \operatorname{div}\mathbf{F}\,dV,$$

or, equivalently,

$$\int\int_S \mathbf{F}\cdot\,d\mathbf{S} = \int\int\int_E \nabla\cdot\mathbf{F}\,dV.$$

This result is known as the *Gauss Divergence Theorem*, or simply the *Divergence Theorem*.

As the Divergence Theorem relates the surface integral of a vector field, known as the *flux* of the vector field through the surface, to an integral of its divergence over a solid, it is quite useful for converting potentially difficult double integrals into triple integrals that may be much easier to evaluate, as the following example demonstrates.

Example (Stewart, Section 13.9, Exercise 6) Let S be the surface of the box with vertices $(\pm 1, \pm 2, \pm 3)$, and let $\mathbf{F}(x, y, z) = \langle x^2z^3, 2xyz^3, xz^4\rangle$. To compute the surface integral of $\mathbf{F}$ over S directly is quite tedious, because S has six faces that must be handled separately. Instead, we apply the Divergence Theorem to integrate $\operatorname{div}\mathbf{F}$ over E, the interior of the box. We then have

$$\begin{aligned}\int\int_S \mathbf{F}\cdot\,d\mathbf{S} &= \int\int\int_E \operatorname{div}\mathbf{F}\,dV\\ &= \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} (x^2z^3)_x + (2xyz^3)_y + (xz^4)_z\,dz\,dy\,dz\\ &= \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} 2xz^3 + 2xz^3 + 4xz^3\,dz\,dy\,dx\end{aligned}$$

$$\begin{aligned} &= \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} 8xz^3\,dz\,dy\,dx \\ &= 32\int_{-1}^{1} x \int_{-3}^{3} z^3\,dz\,dx \\ &= 32\int_{-1}^{1} x \left[\left.\frac{z^4}{4}\right|_{-3}^{3} \right]\,dx \\ &= 0. \end{aligned}$$

□

The Divergence Theorem can also be used to convert a difficult surface integral into an easier one.

Example (Stewart, Section 13.9, Exercise 17) Let $\mathbf{F}(x,y,z) = \langle z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2\rangle$. Let S be the top half of the sphere $x^2 + y^2 + z^2 = 1$. To evaluate the surface integral of $\mathbf{F}$ over S, we note that if we combine S with S_1, the disk $x^2 + y^2 \leq 1$, $z = 0$, with *downward* orientation, we then obtain a new surface S_2 that is the boundary of the top half of the ball $x^2 + y^2 + z^2 \leq 1$, which we denote by E. By the Divergence Theorem,

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} + \int\int_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int\int_{S_2} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_E \operatorname{div}\mathbf{F}\,dV.$$

We parameterize S_1 by

$$x = u\sin v, \quad y = u\cos v, \quad z = 0, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

This parameterization is used instead of the usual one arising from polar coordinates, due to the downward orientation. It follows from

$$\mathbf{r}_u = \langle \sin u, \cos u, 0\rangle, \quad \mathbf{r}_v = \langle u\cos v, -u\sin v, 0\rangle$$

that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, -u\sin^2 v - u\cos^2 v\rangle = u\langle 0, 0, -1\rangle,$$

which points downward, as desired. From

$$\operatorname{div}\mathbf{F}(x,y,z) = (z^2x)_x + \left(\frac{1}{3}y^3 + \tan z\right)_y + (x^2z + y^2)_z = x^2 + y^2 + z^2,$$

which suggests the use of spherical coordinates for the integral over E, we obtain

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} \quad = \quad \int\int\int_E \operatorname{div}\mathbf{F}\,dV - \int\int_{S_1} \mathbf{F}\cdot d\mathbf{S}$$

$$
\begin{aligned}
&= \int\int\int_E (x^2+y^2+z^2)\,dV - \\
&\quad \int_0^1 \int_0^{2\pi} \mathbf{F}(x(u,v),y(u,v),z(u,v)) \cdot u\langle 0,0,-1\rangle\,dv\,du \\
&= \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} \rho^2\rho^2 \sin\phi\,d\phi\,d\theta\,d\rho + \int_0^1 \int_0^{2\pi} u(u^2\cos^2 v)\,dv\,du \\
&= 2\pi \int_0^1 \rho^4 \int_0^{\pi/2} \sin\phi\,d\phi\,d\rho + \int_0^1 u^3 \int_0^{2\pi} \cos^2 v\,dv\,du \\
&= 2\pi \int_0^1 \rho^4 \left[-\cos\phi|_0^{\pi/2} \right] d\rho - \int_0^1 u^3 \int_0^{2\pi} \frac{1+\cos 2v}{2}\,dv\,du \\
&= 2\pi \int_0^1 \rho^4\,d\rho + \int_0^1 u^3 \left[\frac{v}{2} + \frac{\sin 2v}{4} \right]\Bigg|_0^{2\pi} du \\
&= 2\pi \left.\frac{\rho^5}{5}\right|_0^1 + \pi \int_0^1 u^3\,du \\
&= \frac{2\pi}{5} + \pi \left.\frac{u^4}{4}\right|_0^1 \\
&= \frac{2\pi}{5} + \frac{\pi}{4} \\
&= \frac{13\pi}{20}.
\end{aligned}
$$

□

Suppose that $\mathbf{F}$ is a vector field that, at any point, represents the *flow rate* of heat energy, which is the rate of change, with respect to time, of the amount of heat energy flowing through that point. By *Fourier's Law*, $\mathbf{F} = -K\nabla T$, where K is a constant called *thermal conductivity*, and T is a function that indicates temperature.

Now, let E be a three-dimensional solid enclosed by a closed, positively oriented, surface S with *outward* unit normal vector $\mathbf{n}$. Then, by the law of conservation of energy, the *rate of change, with respect to time, of the amount of heat energy inside E is equal to the flow rate, or flux, or heat* **into** *E through S*. That is, if $\rho(x,y,z)$ is the density of heat energy, then

$$\frac{\partial}{\partial t}\int\int\int_E \rho\,dV = \int\int_S \mathbf{F}\cdot(-\mathbf{n})\,dS,$$

where we use $-\mathbf{n}$ because $\mathbf{n}$ is the *outward* unit normal vector, but we need to express the flux *into* E through S.

From the definition of $\mathbf{F}$, and the fact that $\rho = c\rho_0 T$, where c is the *specific heat* and ρ_0 is the mass density, which, for simplicity, we assume to be constant, we have

$$\frac{\partial}{\partial t}\int\int\int_E c\rho_0 T\,dV = \int\int_S K\nabla T\cdot\mathbf{n}\,dS.$$

Next, we note that because c, ρ_0, and E do not depend on time, we can write

$$\int\int\int_E c\rho_0\frac{\partial T}{\partial t}\,dV = \int\int_S K\nabla T\cdot\,d\mathbf{S}.$$

Now, we apply the Divergence Theorem, and obtain

$$\int\int\int_E c\rho_0\frac{\partial T}{\partial t}\,dV = \int\int\int_E K\,\mathrm{div}\,\nabla T\,dV = \int\int\int_E K\nabla^2 T\,dV.$$

That is,

$$\int\int\int_E \left(c\rho_0\frac{\partial T}{\partial t} - K\nabla^2 T\right)\,dV = 0.$$

Since the solid E is arbitrary, it follows that

$$\frac{\partial T}{\partial t} = \frac{K}{c\rho_0}\nabla^2 T.$$

This is known as the *heat equation*, which is one of the most important *partial differential equations* in all of applied mathematics.

3.10 Differential Forms

To date, we have learned the following theorems concerning the evalution of integrals of derivatives:

- The Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

- The Fundamental Theorem of Line Integrals:

$$\int_a^b \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

- Green's Theorem:

$$\int\int_D (Q_x - P_y)\, dA = \int_C P\, dx + Q\, dy$$

- Stokes' Theorem:

$$\int\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

- Gauss' Divergence Theorem:

$$\int\int\int_E \operatorname{div} \mathbf{F}\, dV = \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

All of these theorems relate the integral of the derivative or gradient of a function, or partial derivatives of components of a vector field, over a higher-dimensional region to the integral or sum of the function or vector field over a lower-dimensional region. Now, we will see how the notation of *differential forms* can be used to combine all of these theorems into one. It is *this* notation, as opposed to vectors and operations such as the divergence and curl, that allows the Fundamental Theorem of Calculus to be generalized to functions of several variables.

A *differential form* is an expression consisting of a scalar-valued function $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ and zero or more infinitesimals of the form $dx_1, dx_2, \ldots, dx_n$, where $x_1, x_2, \ldots, x_n$ are the independent variables of f. The *order* of a differential form is defined to be the number of infinitesimals that it includes.

For simplicity, we set $n = 3$ of three variables. With that in mind, a *0-form*, or a differential form of order zero, is simply a scalar-valued function $f(x, y, z)$. A *1-form* is a function $f(x, y, z)$ together with *one* of the expressions dx, dy or dz. A *2-form* is a function $f(x, y, z)$ together with a *pair* of *distinct* infinitesimals, which can be either $dx\, dy$, $dy\, dz$ or $dz\, dx$. Finally, a *3-form* is an expression of the form $f(x, y, z)\, dx\, dy\, dz$.

Example The function $f(x, y, z) = x^2y + y^3z$ is a 0-form on $\mathbb{R}^3$, while $f\, dx = (x^2y + y^3z)\, dx$ and $f\, dy = (x^2y + y^3z)\, dy$ are both examples of a 1-form on $\mathbb{R}^3$. □

Example Let $f(x, y, z) = 1/(x^2 + y^2 + z^2)$. Then $f\, dx\, dy$ is a 2-form on $\mathbb{R}^3 - \{(0, 0, 0\}$, while $f\, dx\, dy\, dz$ is a 3-form on the same domain. □

Forms of the same order can be added and scaled by functions, as the following examples show.

Example Let $f(x, y, z) = e^{x-y} \sin z$ and let $g(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$. Then f, g and $f + g$ are all 0-forms on $\mathbb{R}^3$, and

$$f + g = e^{x-y} \sin z + (x^2 + y^2 + z^2)^{3/2}.$$

That is, addition of 0-forms is identical to addition of functions.

If we define $\omega_1 = f\,dx$ and $\omega_2 = g\,dy$, then ω_1 and ω_2 are both 1-forms on $\mathbb{R}^3$, and so is $\omega = \omega_1 + \omega_2$, where

$$\omega = f\,dx + g\,dy = e^{x-y} \sin z\,dx + (x^2 + y^2 + z^2)^{3/2}\,dy.$$

Furthermore, if $h(x, y, z) = xy^2z^3$, and

$$\eta_1 = f\,dx\,dy, \quad \eta_2 = g\,dz\,dx$$

are 2-forms on $\mathbb{R}^3$, then

$$\eta = h\eta_1 + \eta_2 = xy^2z^3e^{x-y} \sin z\,dx\,dy + (x^2 + y^2 + z^3)^{3/2}\,dz\,dx$$

is also a 2-form on $\mathbb{R}^3$. □

Example Let $f(x, y, z) = \cos x$, $g(x, y, z) = e^y$ and $h(x, y, z) = xyz^2$. Then, $\nu_1 = f\,dx\,dy\,dz$ and $\nu_2 = g\,dx\,dy\,dz$ are 3-forms on $\mathbb{R}^3$, and so is

$$\nu = \nu_1 + h\nu_2 = (\cos x + xyz^2e^y)\,dx\,dy\,dz.$$

□

It should be noted that like addition of functions, addition of differential forms is both commutative, associative, and distributive. Also, there is never any need to add forms of different order, such as adding a 0-form to a 1-form.

We now define two essential operations on differential forms. The first is called the *wedge product*, a multiplication operation for differential forms. Given a k-form ω and an l-form η, where $0 \leq k + l \leq 3$, the wedge product of ω and η, denoted by $\omega \wedge \eta$, is a $(k+l)$-form. It satisfies the following laws:

1. For each k there is a k-form 0 such that $\eta \wedge 0 = 0 \wedge \eta = 0$ for any l-form η.

2. *Distributitivy*: If f is a 0-form, then

$$(f\omega_1 + \omega_2) \wedge \eta = f(\omega_1 \wedge \eta) + (\omega_2 \wedge \eta).$$

3. *Anticommutativity*:
$$\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega).$$

4. *Associativity*:
$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

5. *Homogeneity*: If f is a 0-form, then
$$\omega \wedge (f\eta) = (f\omega) \wedge \eta = f(\omega \wedge \eta).$$

6. If dx_i is a basic 1-form, then $dx_i \wedge dx_i = 0$.

7. If f is a 0-form, then $f \wedge \omega = f\omega$.

Example Let $\omega = f\,dx$ and $\eta = g\,dy$ be 1-forms. Then
$$\omega \wedge \eta = (f\,dx \wedge g\,dy) = fg(dx \wedge dy) = fg\,dx\,dy,$$
by homogeneity, while
$$\eta \wedge \omega = (-1)^{1(1)}(\omega \wedge \eta) = -fg\,dx\,dy.$$
On the other hand, if $\nu = h\,dy\,dz$ is a 2-form, then
$$\nu \wedge \omega = fh(dy\,dz \wedge dx) = fh\,dy\,dz\,dx = -fh\,dy\,dx\,dz = fh\,dx\,dy\,dz$$
by homogeneity and anticommutativity, while
$$\nu \wedge \eta = fh(dy\,dz \wedge dy) = fh\,dy\,dz\,dy = -fh\,dy\,dy\,dz = 0.$$
□

Note that if any 3-form on $\mathbb{R}^3$ is multiplied by a k-form, where $k > 0$, then the result is zero, because there cannot be distinct basic 1-forms in the wedge product of such forms.

Example Let $\omega = x\,dx - y\,dy$, and $\eta = z\,dy\,dz - x\,dz\,dx$. Then
$$\begin{aligned}
\omega \wedge \eta &= (x\,dx - y\,dy) \wedge (z\,dy\,dz - x\,dz\,dx) \\
&= (x\,dx \wedge z\,dy\,dz) - (y\,dy \wedge z\,dy\,dz) - (x\,dx \wedge x\,dz\,dz) + \\
&\quad (y\,dy \wedge x\,dz\,dx) \\
&= xz\,dx\,dy\,dz - yz\,dy\,dy\,dz - x^2\,dx\,dz\,dx + xy\,dy\,dz\,dx \\
&= xz\,dx\,dy\,dz - yz\,dy\,dy\,dz + x^2\,dx\,dx\,dz + xy\,dy\,dz\,dx \\
&= xz\,dx\,dy\,dz - 0 - 0 - xy\,dy\,dx\,dz \\
&= (xz + xy)\,dx\,dy\,dz.
\end{aligned}$$

□

The second operation is differentiation. Given a k-form ω, where $k < 3$, the derivative of ω, denoted by $d\omega$, is a $(k+1)$-form. It satisfies the following laws:

1. If f is a 0-form, then
$$df = f_x\,dx + f_y\,dy + f_z\,dz$$
2. *Linearity*: If ω_1 and ω_2 are k-forms, then
$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$$
3. *Product Rule*: If ω is a k-form and η is an l-form, then
$$d(\omega \wedge \eta) = (d\omega \wedge \eta) + (-1)^k(\omega \wedge d\eta)$$
4. The second derivative of a form is zero; that is, for any k-form ω, $d(d\omega) = 0$.

We now illustrate the use of these differentiation rules.

Example Let $\omega = x^2y^3z^4\,dx\,dy$ be a 2-form. Then, by Linearity and the Product Rule,

$$\begin{aligned} d\omega &= [d(x^2y^3z^4) \wedge dx\,dy] + (-1)^0[x^2y^3z^4 \wedge d(dx\,dy)] \\ &= \big[\big((x^2y^3z^4)_x\,dx + (x^2y^3z^4)_y\,dy + (x^2y^3z^4)_z\,dz\big) \wedge dx\,dy\big] + \\ &\quad \big[x^2y^3z^4 \wedge \{(d(dx) \wedge dy) + (-1)^1(dx \wedge d(dy)\}\big] \\ &= \big[\big(2xy^3z^4\,dx + 3x^2y^2z^4\,dy + 4x^2y^3z^3\,dz\big) \wedge dx\,dy\big] + \\ &\quad \big[x^2y^3z^4 \wedge \{(0 \wedge dy) - (dx \wedge 0)\}\big] \\ &= 2xy^3z^4\,dx\,dx\,dy + 3x^2y^2z^4\,dy\,dx\,dy + 4x^2y^3z^3\,dz\,dx\,dy + 0 \\ &= -4x^2y^3z^3\,dx\,dz\,dy \\ &= 4x^2y^3z^3\,dx\,dy\,dz. \end{aligned}$$

In general, differentiating a k-form ω, when $k > 0$, only requires differentiating the coefficient function with respect to the variables that are *not* among any basic 1-forms that are included in ω. In this example, since $\omega = f\,dx\,dy$, we obtain $d\omega = f_z\,dz\,dx\,dy = f_z\,dx\,dy\,dz$. □

We now consider the kind of differential forms that appear in the theorems of vector calculus.

- Let $\omega = f(x,y,z)$ be a 0-form. Then, by the first law of differentiation,

$$d\omega = \nabla f \cdot \langle dx, dy, dz \rangle.$$

If C is a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$, then

$$\begin{aligned} \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \langle x'(t), y'(t), z'(t) \rangle\,dt \\ &= \int_a^b d\omega(\mathbf{r}(t)) \\ &= \int_C d\omega. \end{aligned}$$

It follows from the Fundamental Theorem of Line Integrals that

$$\int_C d\omega = \omega(\mathbf{r}(b)) - \omega(\mathbf{r}(a)).$$

The boundary of C, ∂C, consists of its initial point A and terminal point B. If we define the "integral" of a 0-form ω over this 0-dimensional region by

$$\int_{\partial C} \omega = \omega(B) - \omega(A),$$

which makes sense considering that, intuitively, the numbers 1 and -1 serve as an appropriate "outward unit normal vector" at the terminal and initial points, respectively, then we have

$$\int_C d\omega = \int_{\partial C} \omega.$$

- Let $\omega = P(x,y)\,dx + Q(x,y)\,dy$ be a 1-form. Then

$$\begin{aligned} d\omega &= d[P(x,y)\,dx] + d[Q(x,y)\,dy] \\ &= dP(x,y) \wedge dx - P(x,y) \wedge d(dx) + dQ(x,y) \wedge dy - \\ &\quad Q(x,y) \wedge d(dy) \\ &= (P_x\,dx + P_y\,dy) \wedge dx - 0 + (Q_x\,dx + Q_y\,dy) \wedge dy - 0 \\ &= P_x\,dx\,dx + P_y\,dy\,dx + Q_x\,dx\,dy + Q_y\,dy\,dy \\ &= (Q_y - P_x)\,dx\,dy. \end{aligned}$$

It follows from Green's Theorem that

$$\int_C \omega = \int \int_D d\omega.$$

- If we proceed similarly with a 1-form

$$\omega = \mathbf{F}\cdot\langle dx, dy, dz\rangle = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz,$$

then we obtain

$$\begin{aligned} d\omega &= \operatorname{curl}\mathbf{F}\cdot\langle dy\,dz, dz\,dx, dx\,dy\rangle \\ &= (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_y - P_x)\,dx\,dy. \end{aligned}$$

Let S be a smooth surface parameterized by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v)\in D.$$

Then the (unnormalized) normal vector $\mathbf{r}_u\times\mathbf{r}_v$ is given by

$$\begin{aligned} \mathbf{r}_u\times\mathbf{r}_v &= \langle x_u, y_u, z_u\rangle \times \langle x_v, y_v, z_v\rangle \\ &= \langle y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v\rangle \\ &= \left\langle \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right\rangle. \end{aligned}$$

We then have

$$\begin{aligned} \int\int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \int\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS \\ &= \int\int_D \operatorname{curl}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv \\ &= \int\int_D \left\{ [R_y(\mathbf{r}(u,v)) - Q_z(\mathbf{r}(u,v))]\frac{\partial(y,z)}{\partial(u,v)} + \right. \\ &\quad [P_z(\mathbf{r}(u,v)) - R_x(\mathbf{r}(u,v))]\frac{\partial(z,x)}{\partial(u,v)} + \\ &\quad \left. [Q_x(\mathbf{r}(u,v)) - P_y(\mathbf{r}(u,v))]\frac{\partial(x,y)}{\partial(u,v)} \right\} du\,dv \\ &= \int\int_S (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + \\ &\quad (Q_y - P_x)\,dx\,dy \\ &= \int\int_S d\omega. \end{aligned}$$

If C is the boundary curve of S, and C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a\le t\le b$, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$

$$\begin{aligned} &= \int_a^b \langle P(\mathbf{r}(t)), Q(\mathbf{r}(t)), R(\mathbf{r}(t)) \cdot \langle x'(t), y'(t), z'(t)\rangle \, dt \\ &= \int_a^b \omega(\mathbf{r}(t))\, dt \\ &= \int_C \omega. \end{aligned}$$

It follows from Stokes' Theorem that

$$\int_C \omega = \int\int_S d\omega.$$

- Let $\mathbf{F} = \langle P, Q, R\rangle$. Let ω be the 2-form

$$\omega = P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy.$$

Then

$$\begin{aligned} d\omega &= dP\, dy\, dz + dQ\, dz\, dx + dR\, dx\, dy \\ &= [P_x\, dx + P_y\, dy + P_z\, dz]\, dy\, dz + [Q_x\, dx + Q_y\, dy + Q_z\, dz]\, dz\, dx + \\ &\quad [R_x\, dx + R_y\, dy + R_z\, dz]\, dx\, dy \\ &= P_x\, dx\, dy\, dz + Q_y\, dy\, dz\, dx + R_z\, dz\, dx\, dy \\ &= P_x\, dx\, dy\, dz - Q_y\, dy\, dx\, dz - R_z\, dx\, dz\, dy \\ &= P_x\, dx\, dy\, dz + Q_y\, dx\, dy\, dz + R_z\, dx\, dy\, dz \\ &= \operatorname{div} \mathbf{F}\, dx\, dy\, dz. \end{aligned}$$

Let E be a solid enclosed by a smooth surface S with positive orientation, and let S be parameterized by

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v) \in D.$$

We then have

$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv \\ &= \int\int_D \langle P(\mathbf{r}(u,v)), Q(\mathbf{r}(u,v)), R(\mathbf{r}(u,v))\rangle \cdot \left\langle \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right\rangle du\, dv \\ &= \int\int_D P(\mathbf{r}(u,v))\frac{\partial(y,z)}{\partial(u,v)} + Q(\mathbf{r}(u,v))\frac{\partial(z,x)}{\partial(u,v)} + \end{aligned}$$

$$
\begin{aligned}
& R(\mathbf{r}(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv \\
= & \int\int_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \\
= & \int\int_S \omega.
\end{aligned}
$$

It follows from the Divergence Theorem that

$$\int\int_S \omega = \int\int\int_E d\omega.$$

Putting all of these results together, we obtain the following combined theorem, that is known as the *General Stokes' Theorem*:

> If M is an oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form defined on an open set containing M, then
>
> $$\int_{\partial M} \omega = \int_M d\omega.$$

The importance of this unified theorem is that, unlike the previously stated theorems of vector calculus, this theorem, through the language of differential forms, can be generalized to functions of any number of variables. This is because operations on differential forms are not defined in terms of other operations, such as the cross product, that are limited to three variables. For example, given a 3-form $\omega = f(x,y,z,w)\,dx\,dy\,dw$, its integral over a 3-dimensional, closed, positively oriented hypersurface S embedded in $\mathbb{R}^4$ is equal to the integral of $d\omega$ over the 4-dimensional solid E that is enclosed by S, where $d\omega$ is computed using the previously stated rules for differentiation and multiplication of differential forms.