These notes correspond to Section 12.8 in Stewart and Section 6.2 in Marsden and Tromba.

**Change of Variables in Multiple Integrals**

Recall that in single-variable calculus, if the integral
\[
\int_a^b f(u) \, du
\]
is evaluated by making a change of variable \( u = g(x) \), such that the interval \( \alpha \leq x \leq \beta \) is mapped by \( g \) to the interval \( a \leq u \leq b \), then
\[
\int_a^b f(u) \, du = \int_\alpha^\beta f(g(x))g'(x) \, dx.
\]

The appearance of the factor \( g'(x) \) in the integrand is due to the fact that if we divide \([a, b]\) into \( n \) subintervals \([u_{i-1}, u_i]\) of equal width \( \Delta u = (b - a)/n \), and if we divide \([\alpha, \beta]\) into \( n \) subintervals \([x_{i-1}, x_i]\) of equal width \( \Delta x = (\beta - \alpha)/n \), then
\[
\Delta u = u_i - u_{i-1} = g(x_i) - g(x_{i-1}) = g'(x_i^*)\Delta x,
\]
where \( x_{i-1} \leq x_i^* \leq x_i \). We will now generalize this change of variable to multiple integrals.

For simplicity, suppose that we wish to evaluate the double integral
\[
\int \int_D f(x, y) \, dA
\]
by making a change of variable
\[
x = g(u, v), \quad y = h(u, v), \quad a \leq u \leq b, \quad c \leq v \leq d.
\]
We divide the interval \([a, b]\) into \( n \) subintervals \([u_{i-1}, u_i]\) of equal width \( \Delta u = (b - a)/n \), and we divide \([c, d]\) into \( m \) subintervals \([v_{i-1}, v_i]\) of equal width \( \Delta v = (d - c)/m \). Then, the rectangle \([u_{i-1}, u_i] \times [v_{i-1}, v_i]\) is approximately mapped by \( g \) and \( h \) into a parallelogram with adjacent sides
\[
\mathbf{r}_u = (g(u_i, v_{i-1}) - g(u_{i-1}, v_{i-1}), h(u_i, v_{i-1}) - h(u_{i-1}, v_{i-1})),
\]
\[
\mathbf{r}_v = (g(u_{i-1}, v_i) - g(u_{i-1}, v_{i-1}), h(u_{i-1}, v_i) - h(u_{i-1}, v_{i-1})).
\]
By the Mean Value Theorem, we have
\[ \mathbf{r}_u \approx (g_u(u_{i-1}, v_{i-1}), h_u(u_{i-1}, v_{i-1})) \Delta u, \quad \mathbf{r}_v \approx (g_v(u_{i-1}, v_{i-1}), h_v(u_{i-1}, v_{i-1})) \Delta v. \]

The area of this parallelogram is given by
\[ |\mathbf{r}_u \times \mathbf{r}_v| = \left| \frac{\partial g}{\partial u} \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v} \frac{\partial h}{\partial u} \right| \Delta u \Delta v. \]

It follows that
\[ \int \int_D f(x, y) \, dx \, dy = \int \int_{\tilde{D}} f(g(u, v), h(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv, \]
where \( \tilde{D} = [a, b] \times [c, d] \) is the domain of \( g \) and \( h \), and
\[ \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \]
is the Jacobian of the transformation from \((u, v)\) to \((x, y)\). It is also the determinant of the Jacobian matrix of the vector-valued function that maps \((u, v)\) to \((x, y)\).

**Example** Let \( D \) be the parallelogram with vertices \((0, 0)\), \((2, 4)\), \((6, 1)\), and \((8, 5)\). To integrate a function \( f(x, y) \) over \( D \), we can use a change of variable \((x, y) = (g(u, v), h(u, v))\) that maps a rectangle to this parallelogram, and then integrate over the rectangle.

Using the vertices, we find that the equations of the edges are
\[ -x + 6y = 0, \quad -x + 6y = 22, \quad 2x - y = 0, \quad 2x - y = 11. \]

Therefore, if we define the new variables \( u \) and \( v \) by the equations
\[ u = -x + 6y, \quad v = 2x - y, \]
then, for \((x, y) \in D\), we have \((u, v)\) belonging to the rectangle \(0 \leq u \leq 22, 0 \leq v \leq 11\).

To rewrite an integral over \( D \) in terms of \( u \) and \( v \), it is much easier to express the original variables in terms of the new variables than the other way around. Therefore, we need to solve the equations defining \( u \) and \( v \) for \( x \) and \( y \). From the equation for \( u \), we have \( x = 6y - u \). Substituting into the equation for \( v \), we obtain \( v = 2(6y - u) - y \), which yields \( y = h(u, v) = \frac{1}{11}(2u + v) \). Substituting this into the equation for \( u \) yields \( x = g(u, v) = \frac{1}{11}(u + 6v) \).

The Jacobian of this transformation is
\[ \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{1}{112}[1(1) - 6(2)] = -\frac{1}{11}. \]

We conclude that
\[ \int \int_D f(x, y) \, dx \, dy = \frac{1}{11} \int \int_{\tilde{D}} f(g(u, v), h(u, v)) \, du \, dv. \]
In general, when integrating a function $f(x_1, x_2, \ldots, x_n)$ over a region $D \subset \mathbb{R}^n$, if the integral is evaluated using a change of variable $(x_1, x_2, \ldots, x_n) = g(u_1, u_2, \ldots, u_n)$ that maps a region $E \subset \mathbb{R}^n$ to $D$, then

$$
\int_D f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n = \int_E (f \circ g)(u_1, \ldots, u_n) \mid \det(J_g(u_1, \ldots, u_n)) \mid du_1 \cdots du_n,
$$

where

$$
J_g(u_1, u_2, \ldots, u_n) = \begin{bmatrix}
\frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\
\frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n}
\end{bmatrix}
$$

is the Jacobian matrix of $g$ and $\det(J_g(u_1, u_2, \ldots, u_n))$ is its determinant, which is simply referred to as the Jacobian of the transformation $g$.

**Example** Consider the transformation from spherical to Cartesian coordinates,

$$
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
$$

Then, the Jacobian matrix of this transformation is

$$
\begin{bmatrix}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{bmatrix}
= \begin{bmatrix}
\sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
\rho \cos \phi & 0 & -\sin \phi
\end{bmatrix}.
$$

It follows that the Jacobian of this transformation is given by the determinant of this matrix,

$$
\left|\begin{array}{ccc}
\sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
\cos \phi & 0 & -\sin \phi
\end{array}\right| = \cos \phi \left|\begin{array}{ccc}
-\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\
\rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\
-\rho \sin \phi
\end{array}\right| - \rho \cos \phi \left|\begin{array}{ccc}
\sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\
\sin \phi \sin \theta & \rho \sin \phi \cos \theta \\
\rho \sin \phi \cos \theta
\end{array}\right|
$$

$$
= \cos \phi \left[-\rho^2 \sin \phi \sin \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta\right] -
\rho \sin \phi \left[\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \cos^2 \phi \sin^2 \theta\right]
$$

$$
= -\rho^2 \cos^2 \phi \sin \phi - \rho^2 \sin^2 \phi \sin \phi
$$

$$
= -\rho^2 \sin \phi.
$$

The absolute value of the Jacobian is the factor that must be included in the integrand when converting a triple integral from Cartesian to spherical coordinates. □
Practice Problems

Practice problems from the recommended textbooks are:

- Stewart: Section 12.8, Exercises 1-15 odd
- Marsden/Tromba: Section 6.2, Exercises 1, 3, 5