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**Lecture 18 Notes**

These notes correspond to Section 13.3 in Stewart and Section 2.4 and 7.2 in Marsden and Tromba.

## The Fundamental Theorem for Line Integrals

We have learned that the line integral of a vector field  $\mathbf{F}$  over a curve piecewise smooth  $C$ , that is parameterized by a vector-valued function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Now, suppose that  $\mathbf{F}$  continuous, and is a *conservative* vector field; that is,  $\mathbf{F} = \nabla f$  for some scalar-valued function  $f$ . Then, by the Chain Rule, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \frac{d}{dt}[(f \circ \mathbf{r})(t)] dt = (f \circ \mathbf{r})(t)|_a^b = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

This is the *Fundamental Theorem of Line Integrals*, which is a generalization of the Fundamental Theorem of Calculus.

If the curve  $C$  is a *closed* curve; that is, the initial and terminal points of  $C$  are the same, then  $\mathbf{r}(b) = \mathbf{r}(a)$ , which yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0.$$

If we decompose  $C$  into two curves  $C_1$  and  $C_2$ , and use the fact that the sign of the line integral of a vector field over a curve depends on the orientation of the curve, then we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0.$$

That is,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

However,  $C_1$  and  $-C_2$  have the same initial and terminal points. It follows that if  $\mathbf{F}$  is conservative within an open, *connected* domain  $D$  (so that any two points in  $D$  can be connected by a path that lies within  $D$ ), then the line integral of  $\mathbf{F}$  is *independent of path* in  $D$ ; that is, the value of the line integral of  $\mathbf{F}$  over a path  $C$  depends only on its initial and terminal points.

The *converse* of this statement is also true: if the line integral of a vector field  $\mathbf{F}$  is independent of path within an open, connected domain  $D$ , then  $\mathbf{F}$  is a conservative vector field on  $D$ . To see this, we consider the two-variable case and let  $D$  be a region in  $\mathbb{R}^2$ . Furthermore, we let  $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ . We choose an arbitrary point  $(a, b) \in D$ , and define

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}.$$

Since this line integral is independent of path, we can define  $f(x, y)$  using any path between  $(a, b)$  and  $(x, y)$  that we choose, knowing that its value at  $(x, y)$  will be the same in any case.

By choosing a path that ends with a horizontal line segment from  $(x_1, y)$  to  $(x, y)$  contained entirely in  $D$ , parametrized by  $x = t$ ,  $y = y$ , for  $x_1 \leq t \leq x$ , we can show that

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x} \left[ \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} \right] + \frac{\partial}{\partial x} \left[ \int_{(x_1,y)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} \right] \\ &= 0 + \frac{\partial}{\partial x} \left[ \int_{x_1}^x P(x(t), y) x'(t) dt + Q(x(t), y) y'(t) dt \right] \\ &= \frac{\partial}{\partial x} \left[ \int_{x_1}^x P(t, y) dt + 0 \right] \\ &= P(x, y). \end{aligned}$$

Using a similar argument, we can show that  $\partial f / \partial y = Q$ . We have thus shown that  $\mathbf{F}$  is conservative, and conclude that  $\mathbf{F}$  is a conservative vector field if and only if its line integral is independent of path.

However, in order to use the Fundamental Theorem of Line Integrals to evaluate the line integral of a conservative vector field, it is necessary to obtain the function  $f$  such that  $\nabla f = F$ . Furthermore, the theorem cannot be applied to a vector field that is not conservative, so we need to be able to confirm that a given vector field is conservative before we proceed.

Continuing to restrict ourselves to the two-variable case, suppose that  $\mathbf{F} = \langle P, Q \rangle$  is a conservative vector field defined on a domain  $D$ , and that  $P$  and  $Q$  have continuous first partial derivatives. Then, we have

$$\frac{\partial f}{\partial x} = P, \quad \frac{\partial f}{\partial y} = Q,$$

for some function  $f$ . It follows that

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

However, by Clairaut's Theorem, these mixed second partial derivatives of  $f$  are equal, so it follows that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

if  $\mathbf{F} = \langle P, Q \rangle$  is conservative.

If the domain  $D$  is *simply connected*, meaning that *any* region enclosed by a closed curve in  $D$  contains only points in  $D$  (informally,  $D$  has “no holes”), then the converse is true: if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

in  $D$ , then  $\mathbf{F} = \langle P, Q \rangle$  is a conservative vector field. Similarly, if  $\mathbf{F} = \langle P, Q, R \rangle$  is a vector field defined on a simply connected domain  $D \subseteq \mathbb{R}^3$ , and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

then  $\mathbf{F}$  is conservative.

It remains to be able to find the function  $f$  such that  $\nabla f = \mathbf{F}$  for a given vector field  $\mathbf{F} = \langle P, Q \rangle$  that is known to be conservative. The general technique is as follows:

- Integrate  $P(x, y)$  with respect to  $x$  to obtain

$$f(x, y) = f_1(x, y) + g(y),$$

where  $f_1(x, y)$  is obtained by anti-differentiation of  $P(x, y)$ , and  $g(y)$  is an unknown function that plays the role of the constant of integration, since  $f(x, y)$  is obtained by anti-differentiating with respect to  $x$ .

- Differentiate  $f$  with respect to  $y$  to obtain

$$\frac{\partial}{\partial y}[f_1(x, y)] + g'(y) = Q(x, y),$$

and solve for  $g'(y)$ .

- Integrate  $g'(y)$  with respect to  $y$  to complete the definition of  $f(x, y)$ , up to a constant of integration.

A similar procedure can be used for a vector field defined on  $\mathbb{R}^3$ , except that the function  $g$  depends on both  $y$  and  $z$ , and differentiation with respect to both  $y$  and  $z$  is needed to completely define the function  $f(x, y, z)$  such that  $\nabla f = \mathbf{F}$ .

**Example** (Stewart, Section 13.3, Exercise 14) Let

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle.$$

To confirm that  $\mathbf{F}$  is conservative, we check the appropriate first partial derivatives of  $P$ ,  $Q$  and  $R$ :

$$P_y = 2y = Q_x, \quad P_z = 2x = R_x, \quad Q_z = 0 = R_y.$$

Now, to find a function  $f(x, y, z)$  such that  $\nabla f = \mathbf{F}$ , which must satisfy  $f_x = P$ , we integrate  $P(x, y, z)$  with respect to  $x$  and obtain

$$f(x, y, z) = x^2z + y^2x + g(y, z).$$

Differentiating with respect to  $y$  and  $z$  yields the equations

$$\begin{aligned} f_y(x, y, z) &= 2xy + g_y(y, z) = Q(x, y, z) = 2xy, \\ f_z(x, y, z) &= x^2 + g_z(y, z) = R(x, y, z) = x^2 + 3z^2. \end{aligned}$$

It follows that

$$g_y(y, z) = 0, \quad g_z(y, z) = 3z^2,$$

which yields

$$g(y, z) = z^3 + K$$

for some constant  $K$ . We conclude that  $\mathbf{F} = \nabla f$  where

$$f(x, y, z) = x^2z + y^2x + z^3 + K$$

where  $K$  is an arbitrary constant.

To evaluate the line integral of  $\mathbf{F}$  over the curve  $C$  parametrized by

$$x = t^2, \quad y = t + 1, \quad z = 2t - 1, \quad 0 \leq t \leq 1,$$

we apply the Fundamental Theorem of Line Integrals and obtain

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(x(1), y(1), z(1)) - f(x(0), y(0), z(0)) \\ &= f(1, 2, 1) - f(0, 1, -1) \\ &= 1^2(1) + 2^2(1) + 1^3 + K - (0^2(-1) + 1^2(0) + (-1)^3 + K) \\ &= 1 + 4 + 1 + K - (0 + 0 - 1 + K) \\ &= 7. \end{aligned}$$

□

Let  $\mathbf{F}$  represent a force field. Then, recall that the work done by the force field to move an object along a path  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ , is given by the line integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

From Newton's Second Law of Motion, we have

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t),$$

where  $m$  is the mass of the object, and  $\mathbf{r}''(t) = \mathbf{a}(t)$  is its acceleration. We then have

$$\begin{aligned} W &= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt \\ &= \frac{1}{2}m \int_a^b \frac{d}{dt}[\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt \\ &= \frac{1}{2}m \int_a^b \frac{d}{dt}[\|\mathbf{r}'(t)\|^2] dt \\ &= \frac{1}{2}m\|\mathbf{v}(b)\|^2 - \frac{1}{2}m\|\mathbf{v}(a)\|^2 \end{aligned}$$

where  $\mathbf{v}(t) = \mathbf{r}'(t)$  is the velocity of the object.

It follows that

$$W = K(B) - K(A),$$

where  $A = \mathbf{r}(a)$  and  $B = \mathbf{r}(b)$  are the initial and terminal points, respectively, and

$$K(P) = \frac{1}{2}m\|\mathbf{v}(t)\|^2, \quad \mathbf{r}(t) = P,$$

is the *kinetic energy* of the object at the point  $P$ . That is, the work done by the force field along  $C$  is the change in the kinetic energy from point  $A$  to point  $B$ .

If  $\mathbf{F}$  is also a conservative force field, then  $\mathbf{F} = -\nabla P$ , where  $P$  is the *potential energy*. It follows from the Fundamental Theorem of Line Integrals that

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = - \int_C \nabla P \cdot d\mathbf{r} = -[P(B) - P(A)].$$

We conclude that

$$P(A) + K(A) = P(B) + K(B).$$

That is, when an object is moved by a conservative force field, then its total energy remains constant. This is known as the *Law of Conservation of Energy*.

## Practice Problems

Practice problems from the recommended textbooks are:

- Stewart: Section 13.3, Exercises 3-19 odd
- Marsden/Tromba: Section 7.2, Exercises 15, 17