Surface Integrals

Surface Integrals of Scalar-Valued Functions

Previously, we have learned how to integrate functions along curves. If a smooth space curve $C$ is parameterized by a function $r(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$, then the arc length $L$ of $C$ is given by the integral

$$\int_a^b \| r'(t) \| \, dt.$$ 

Similarly, the integral of a scalar-valued function $f(x, y, z)$ along $C$ is given by

$$\int_C f \, ds = \int_a^b f(x(t), y(t), z(t)) \| r'(t) \| \, dt.$$ 

It follows that the integral of $f(x, y, z) \equiv 1$ along $C$ is equal to the arc length of $C$.

We now define integrals of scalar-valued functions on surfaces in an analogous manner. Recall that the area of a smooth surface $S$, parametrized by $r(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ for $(u, v) \in D$, is given by the integral

$$A(S) = \int_D \| r_u \times r_v \| \, du \, dv.$$ 

To integrate a scalar-valued function $f(x, y, z)$ over $S$, we assume for simplicity that $D$ is a rectangle, and divide it into sub-rectangles $\{R_{ij}\}$ of dimension $\Delta u$ and $\Delta v$, as we did when we derived the formula for $A(S)$. Then, the function $r$ maps each sub-rectangle $R_{ij}$ into a surface patch $S_{ij}$ that has area $\Delta S_{ij}$. This area is then multiplied by $f(P_{ij}^*)$, where $P_{ij}^*$ is any point on $S_{ij}$.

Letting $\Delta u, \Delta v \to 0$, we obtain the surface integral of $f$ over $S$ to be

$$\int_S f(x, y, z) \, dS = \lim_{\Delta u, \Delta v \to 0} \sum_{i=1}^n \sum_{j=1}^m f(P_{ij}^*) \Delta S_{ij} = \int_D f(r(u, v)) \| r_u \times r_v \| \, du \, dv,$$

since, in the limit as $\Delta u, \Delta v \to 0$, we have

$$\Delta S_{ij} \to \| r_u \times r_v \| \, \Delta u \, \Delta v.$$ 

Note that if $f(x, y, z) \equiv 1$, then the surface integral of $f$ over $S$ yields the area of $S$, $A(S)$. 

Example (Stewart, Section 13.7, Exercise 6) Let $S$ be the helicoid with parameterization

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq \pi.$$ 

Then we have

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle,$$

which yields

$$||\mathbf{r}_u \times \mathbf{r}_v|| = ||\langle \sin v, -\cos v, u \rangle|| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}.$$ 

It follows that

$$\int \int_S \sqrt{1 + x^2 + y^2} \, dS = \int_0^1 \int_0^\pi \sqrt{1 + (u \cos v)^2 + (u \sin v)^2} ||\mathbf{r}_u \times \mathbf{r}_v|| \, dv \, du$$

$$= \int_0^1 \int_0^\pi \sqrt{1 + u^2} \sqrt{1 + u^2} \, dv \, du$$

$$= \int_0^1 \int_0^\pi \sqrt{1 + u^2} \, dv \, du$$

$$= \pi \int_0^1 \sqrt{1 + u^2} \, du$$

$$= \pi \left( u + \frac{u^3}{3} \right) \bigg|_0^1$$

$$= \frac{4\pi}{3}.$$ 

□

The surface integral of a scalar-valued function is useful for computing the mass and center of mass of a thin sheet. If the sheet is shaped like a surface $S$, and it has density $\rho(x, y, z)$, then the mass is given by the surface integral

$$m = \int \int_S \rho(x, y, z) \, dS,$$

and the center of mass is the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \int \int_S x\rho(x, y, z) \, dS, \quad \bar{y} = \frac{1}{m} \int \int_S y\rho(x, y, z) \, dS, \quad \bar{z} = \frac{1}{m} \int \int_S z\rho(x, y, z) \, dS.$$
Surface Integrals of Vector Fields

Let \( \mathbf{v} \) be a vector field defined on \( \mathbb{R}^3 \) that represents the velocity field of a fluid, and let \( \rho \) be the density of the fluid. Then, the rate of flow of the fluid, which is defined to be the rate of change with respect to time of the amount of fluid (mass), per unit area, is given by \( \rho \mathbf{v} \).

To determine the total amount of fluid that is crossing \( S \), called the flux across \( S \), we divide \( S \) into several small patches \( S_{ij} \), as we did when we defined the surface integral of a scalar-valued function. Since each patch \( S_{ij} \) is approximately planar (that is, parallel to a plane), we can approximate the flux across \( S_{ij} \) by

\[
(\rho \mathbf{v} \cdot \mathbf{n}) A(S_{ij}),
\]

where \( \mathbf{n} \) is a unit vector that is normal (perpendicular) to \( S_{ij} \). This is because if \( \theta \) is the angle between \( S_{ij} \) and the direction of \( \mathbf{v} \), then the fluid directed at \( S_{ij} \) is effectively passing through a region of area \( A(S_{ij})|\cos \theta| \).

If we sum the flux over each patch, and let the areas of the patches approach zero, then we obtain the total flux across \( S \),

\[
\int \int_S \rho(x,y,z)\mathbf{v}(x,y,z) \cdot \mathbf{n}(x,y,z) dS,
\]

where \( \mathbf{n}(x,y,z) \) is a continuous function that describes a unit normal vector at each point \((x,y,z)\) on \( S \). For a general vector field \( \mathbf{F} \), we define the surface integral of \( \mathbf{F} \) over \( S \) by

\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} dS.
\]

When \( \mathbf{F} \) represents an electric field, we call the surface integral of \( \mathbf{F} \) over \( S \) the electric flux of \( \mathbf{F} \) through \( S \). Alternatively, if \( \mathbf{F} = -K \nabla u \), where \( u \) is a function that represents temperature and \( K \) is a constant that represents thermal conductivity, then the surface integral of \( \mathbf{F} \) over a surface \( S \) is called the heat flow or heat flux across \( S \).

If \( S \) is parameterized by a function \( \mathbf{r}(u,v) \), where \((u,v) \in D\), then

\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||},
\]

and we then have

\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} ||\mathbf{r}_u \times \mathbf{r}_v|| dA
\]

\[
= \int \int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA.
\]
This is analogous to the definition of the line integral of a vector field over a curve \( C \),
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
\]

Just as the orientation of a curve was relevant to the line integral of a vector field over a curve, the orientation of a surface is relevant to the surface integral of a vector field. We say that a surface \( S \) is orientable, or oriented, if, at each point \((x, y, z)\) in \( S \), it is possible to choose a unique vector \( \mathbf{n}(x, y, z) \) that is normal to the tangent plane of \( S \) at \((x, y, z)\), in such a way that \( \mathbf{n}(x, y, z) \) varies continuously over \( S \). The particular choice of \( \mathbf{n} \) is called an orientation.

An orientable surface has two orientations, or, informally, two “sides”, with normal vectors \( \mathbf{n} \) and \(-\mathbf{n}\). This definition of orientability excludes the Möbius strip, because for this surface, it is possible for a continuous variation of \((x, y, z)\) to yield two distinct normal vectors at every point of the surface, that are negatives of one another. Geometrically, the Möbius strip can be said to have only one “side”, because negating any choice of continuously varying \( \mathbf{n} \) yields the same normal vectors.

For a surface that is the graph of a function \( z = g(x, y) \), if we choose the parametrization
\[
x = u, \quad y = v, \quad z = g(u, v),
\]
then from
\[
\mathbf{r}_u = \langle 1, 0, g_u \rangle, \quad \mathbf{r}_v = \langle 0, 1, g_v \rangle,
\]
we obtain
\[
\mathbf{r}_u \times \mathbf{r}_v = \langle -g_u, -g_v, 1 \rangle = \langle -g_x, -g_y, 1 \rangle
\]
which yields
\[
\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{1 + g_x^2 + g_y^2}}.
\]
Because the \( z \)-component of this vector is positive, we call this choice of \( \mathbf{n} \) an upward orientation of the surface, while \(-\mathbf{n}\) is a downward orientation.

**Example** (Stewart, Section 13.7, Exercise 22) Let \( S \) be the part of the cone \( z = \sqrt{x^2 + y^2} \) that lies beneath the plane \( z = 1 \), with downward orientation. We wish to evaluate the surface integral
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S}
\]
where \( \mathbf{F} = \langle x, y, z^4 \rangle \).

First, we must compute the unit normal vector for \( S \). Using cylindrical coordinates yields the parameterization
\[
x = u \cos v, \quad y = u \sin v, \quad z = u, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.
\]
We then have
\[ \mathbf{r}_u = (\cos v, \sin v, 1), \quad \mathbf{r}_v = (-u \sin v, u \cos v, 0), \]
which yields
\[ \mathbf{r}_u \times \mathbf{r}_v = (-u \cos v, -u \sin v, u \cos^2 v + u \sin^2 v) = u(-\cos v, -\sin v, 1). \]
Because we assume downward orientation, we must have the z-component of the normal vector be negative. Therefore, \( \mathbf{r}_u \times \mathbf{r}_v \) must be negated, which yields
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = -\int \int_D \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA,
\]
where \( D \) is the domain of the parameters \( u \) and \( v \), the rectangle \([0, 1] \times [0, 2\pi]\). Evaluating this integral, we obtain
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = -\int \int_D \langle u \cos v, u \sin v, u^4 \rangle \cdot u(-\cos v, -\sin v, 1) \, dA \\
= -\int_0^{2\pi} \int_0^1 (-u \cos^2 v - u \sin^2 v + u^4) u \, du \, dv \\
= \int_0^{2\pi} \int_0^1 (u^2 - u^5) \, du \, dv \\
= 2\pi \left[ \frac{1}{3} - \frac{1}{6} \right] \\
= \frac{\pi}{3}.
\]
An alternative approach is to retain Cartesian coordinates, and then use the formula for the unit normal for a downward orientation of a surface that is the graph of a function \( z = g(x,y) \),
\[
\mathbf{n} = -\frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{g_x^2 + g_y^2 + 1}} = \frac{1}{\sqrt{2}} \left[ \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right].
\]
This approach still requires a conversion to polar coordinates to integrate over the unit disk in the \( xy \)-plane. \( \square \)

For a closed surface \( S \), which is the boundary of a solid region \( E \), we define the positive orientation of \( S \) to be the choice of \( \mathbf{n} \) that consistently point outward from \( E \), while the inward-pointing normals define the negative orientation.
Example (Stewart, Section 13.7, Exercise 26) To evaluate the surface integral
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S}
\]
where \( \mathbf{F}(x, y, z) = \langle y, z - y, x \rangle \) and \( S \) is the surface of the tetrahedron with vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\), we must evaluate surface integrals over each of the four faces of the tetrahedron separately. We assume positive (outward) orientation.

For the first side, \( S_1 \), with vertices \((0, 0, 0), (1, 0, 0) \) and \((0, 1, 0)\), we first parameterize the side using
\[
x = u, \quad y = 0, \quad z = v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u.
\]
Then, from
\[
\mathbf{r}_u = \langle 1, 0, 0 \rangle, \quad \mathbf{r}_v = \langle 0, 0, 1 \rangle,
\]
we obtain
\[
\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -1, 0 \rangle.
\]
This vector is pointing outside the tetrahedron, so it is the outward normal vector that we wish to use. Therefore, the surface integral of \( \mathbf{F} \) over \( S_1 \) is
\[
\int \int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-u} \langle 0, v - 0, u \rangle \cdot \langle 0, -1, 0 \rangle \, dv \, du
\]
\[
= - \int_0^1 \int_0^{1-u} v \, dv \, du
\]
\[
= - \int_0^1 \frac{v^2}{2} \bigg|_0^{1-u} \, du
\]
\[
= - \frac{1}{2} \int_0^1 (1 - u)^2 \, du
\]
\[
= \frac{1}{2} \left( \frac{(1 - u)^3}{3} \right) \bigg|_0^1
\]
\[
= \frac{1}{6}.
\]

For the second side, \( S_2 \), with vertices \((0, 0, 0), (0, 1, 0) \) and \((0, 0, 1)\), we parameterize using
\[
x = 0, \quad y = u, \quad z = v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u.
\]
Then, from
\[
\mathbf{r}_u = \langle 0, 1, 0 \rangle, \quad \mathbf{r}_v = \langle 0, 0, 1 \rangle,
\]
we obtain
\[
\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 0, 0 \rangle.
\]
This vector is pointing inside the tetrahedron, so we must negate it to obtain the outward normal vector. Therefore, the surface integral of $\mathbf{F}$ over $S_2$ is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-u} (u, v - u, 0) \cdot (-1, 0, 0) \, dv \, du$$

$$= - \int_0^1 \int_0^{1-u} u \, dv \, du$$

$$= \int_0^1 u(u - 1) \, du$$

$$= \left( \frac{u^3}{3} - \frac{u^2}{2} \right) \bigg|_0^1$$

$$= -\frac{1}{6}.$$

For the base $S_3$, with vertices $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$, we parametrize using

\[ x = u, \quad y = v, \quad z = 0, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u. \]

Then, from

\[ \mathbf{r}_u = (1, 0, 0), \quad \mathbf{r}_v = (0, 1, 0), \]

we obtain

\[ \mathbf{r}_u \times \mathbf{r}_v = (0, 0, 1). \]

This vector is pointing inside the tetrahedron, so we must negate it to obtain the outward normal vector. Therefore, the surface integral of $\mathbf{F}$ over $S_3$ is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-u} (v, 0 - v, u) \cdot (0, 0, -1) \, dv \, du$$

$$= - \int_0^1 \int_0^{1-u} u \, dv \, du$$

$$= \int_0^1 u(u - 1) \, du$$

$$= \left( \frac{u^3}{3} - \frac{u^2}{2} \right) \bigg|_0^1$$

$$= -\frac{1}{6}.$$

Finally, for the “top” face $S_4$, with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, we parametrize using

\[ x = u, \quad y = v, \quad z = 1 - u - v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u, \]
since the equation of the plane containing this face is \( x + y + z - 1 = 0 \). This can be determined by using the three vertices to obtain two vectors within the plane, and then computing their cross product to obtain the plane’s normal vector.

Then, from
\[
\mathbf{r}_u = (1, 0, -1), \quad \mathbf{r}_v = (0, 1, -1),
\]
we obtain
\[
\mathbf{r}_u \times \mathbf{r}_v = (1, 1, 1).
\]
This vector is pointing outside the tetrahedron, so it is the outward normal vector that we wish to use. Therefore, the surface integral of \( \mathbf{F} \) over \( S_4 \) is
\[
\int \int_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-u} \langle v, 1 - u - 2v, u \rangle \cdot (1, 1, 1) \, dv \, du
\]
\[
= \int_0^1 \int_0^{1-u} (1 - v) \, dv \, du
\]
\[
= \int_0^1 \left( v - \frac{v^2}{2} \right) \bigg|_0^{1-u} \, du
\]
\[
= \int_0^1 1 - u - \frac{(1 - u)^2}{2} \, du
\]
\[
= \int_0^1 \frac{1}{2} - \frac{1}{2} u^2 \, du
\]
\[
= \left( \frac{u}{2} - \frac{u^3}{6} \right) \bigg|_0^1
\]
\[
= \frac{1}{3}.
\]
Adding the four integrals together yields
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{3} = -\frac{1}{6}.
\]
\(\square\)
Practice Problems

Practice problems from the recommended textbooks are:

- Stewart: Section 13.7, Exercises 5-27 odd
- Marsden/Tromba: Section 7.5, Exercises 1, 3, 7; Section 7.6, Exercises 3-9 odd