These notes correspond to Lesson 25 in the text.

**The Finite Fourier Transforms**

When solving a PDE on a finite interval $0 < x < L$, whether it be the heat equation or wave equation, it can be very helpful to use a finite Fourier transform. In particular, we have the finite sine transform

$$S_n = S[f] = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, dx, \quad n = 1, 2, \ldots,$$

with its inverse sine transform

$$S^{-1}[S_n] = f(x) = \sum_{n=1}^{\infty} S_n \sin(n\pi x/L).$$

This transform should be used with Dirichlet boundary conditions, that specify the value of $u$ at $x = 0$ and $x = L$.

When Neumann boundary conditions are used, that specify the value of $u_x$ at $x = 0$ and $x = L$, it is best to use the finite cosine transform

$$C_n = C[f] = \frac{2}{L} \int_0^L f(x) \cos(n\pi x/L) \, dx, \quad n = 0, 1, 2, \ldots,$$

with its inverse sine transform

$$C^{-1}[C_n] = f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\pi x/L).$$

Both of these transforms can be used to reduce a PDE to an ODE.

**Examples of the Sine Transform**

Consider the function $f(x) = 1$ on $(0, 1)$. If we apply the finite sine transform to this function, we obtain

$$S_n = 2 \int_0^1 \sin(n\pi x) \, dx$$

$$= -\frac{2}{n\pi} \cos(n\pi x) \bigg|_0^1$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

Applying the inverse sine transform yields

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x).$$
Properties of the Transforms

To apply these transforms to PDEs, we must know how to transform appropriate derivatives. We have the following rules:

\[
\begin{align*}
S[u_t] &= \frac{dS[u]}{dt}, & S[u_{tt}] &= \frac{d^2S[u]}{dt^2}, \\
C[u_t] &= \frac{dC[u]}{dt}, & C[u_{tt}] &= \frac{d^2C[u]}{dt^2}, \\
S[u_{xx}] &= -\left[\frac{n\pi}{L}\right]^2 S[u] + \frac{2n\pi}{L^2} [u(0, t) + (-1)^{n+1} u(L, t)], \\
C[u_{xx}] &= -\left[\frac{n\pi}{L}\right]^2 C[u] - \frac{2}{L} [u_x(0, t) + (-1)^{n+1} u_x(L, t)].
\end{align*}
\]

The last two rules can be obtained by applying integration by parts twice.

Solving Problems via Finite Transforms

We illustrate the use of finite Fourier transforms by solving the IBVP

\[
\begin{align*}
u_{tt} &= u_{xx} + \sin(\pi x), & 0 < x < 1, & t > 0, \\
u(0, t) &= 0, & u(1, t) &= 0, & t > 0, \\
u(x, 0) &= 1, & u_t(x, 0) &= 0, & 0 < x < 1.
\end{align*}
\]

Because this problem has Dirichlet boundary conditions, we use the finite sine transform. From the preceding example, the transform of the initial conditions are

\[
S_n(0) = \left\{ \begin{array}{ll}
\frac{4}{n\pi} & n \text{ odd} \\
0 & n \text{ even}
\end{array} \right., \quad S_n'(0) = 0.
\]

Using the definition and aforementioned properties, we obtain the transform of the PDE,

\[
\begin{align*}
S_1''(t) &= -\pi^2 S_1(t) + 4, \\
S_n''(t) &= -(n\pi)^2 S_n(t), & n &= 2, 3, \ldots.
\end{align*}
\]

The ODE for \(S_1(t)\) is nonhomogeneous, and can be solved using either the method of undetermined coefficients or variation of parameters. The general solution is

\[
S_1(t) = A \cos(\pi t) + B \sin(\pi t) + C,
\]

where \(A, B\) and \(C\) are constants. Substituting this form of the solution into the ODE and initial conditions yields

\[
S_1(t) = \left(\frac{4}{\pi} - \frac{1}{\pi^2}\right) \cos(\pi t) + \frac{1}{\pi^2}.
\]

The ODEs for \(S_n(t), n > 1\), are homogeneous and can easily be solved to obtain

\[
S_n(t) = \left\{ \begin{array}{ll}
\frac{4}{n\pi} \cos(n\pi t) & n = 3, 5, 7, \ldots, \\
0 & n = 2, 4, 6, \ldots
\end{array} \right.
\]

Applying the inverse sine transform, we conclude that the solution is

\[
u(x, t) = \left[\left(\frac{4}{\pi} - \frac{1}{\pi^2}\right) \cos(\pi t) + \frac{1}{\pi^2}\right] \sin(\pi x) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n + 1} \cos((2n + 1)\pi t) \sin((2n + 1)\pi x).
\]