Romberg Integration

Richardson extrapolation is not only used to compute more accurate approximations of derivatives, but is also used as the foundation of a numerical integration scheme called Romberg integration. In this scheme, the integral

\[ I(f) = \int_a^b f(x) \, dx \]

is approximated using the Composite Trapezoidal Rule with step sizes \( h_k = (b - a)2^{-k} \), where \( k \) is a nonnegative integer. Then, for each \( k \), Richardson extrapolation is used \( k - 1 \) times to previously computed approximations in order to improve the order of accuracy as much as possible.

More precisely, suppose that we compute approximations \( T_{1,1} \) and \( T_{2,1} \) to the integral, using the Composite Trapezoidal Rule with one and two subintervals, respectively. That is,

\[
T_{1,1} = \frac{b - a}{2} [f(a) + f(b)] \\
T_{2,1} = \frac{b - a}{4} \left[ f(a) + 2f\left(\frac{a + b}{2}\right) + f(b) \right].
\]

Suppose that \( f \) has continuous derivatives of all orders on \([a, b]\). Then, the Composite Trapezoidal Rule, for a general number of subintervals \( n \), satisfies

\[
\int_a^b f(x) \, dx = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + \sum_{i=1}^{\infty} K_i h^{2i},
\]

where \( h = (b - a)/n \), \( x_j = a + jh \), and the constants \( \{K_i\}_{i=1}^{\infty} \) depend only on the derivatives of \( f \).

It follows that we can use Richardson Extrapolation to compute an approximation with a higher order of accuracy. If we denote the exact value of the integral by \( I(f) \) then we have

\[
T_{1,1} = I(f) + K_1 h^2 + O(h^4) \\
T_{2,1} = I(f) + K_1 (h/2)^2 + O(h^4)
\]

Neglecting the \( O(h^4) \) terms, we have a system of equations that we can solve for \( K_1 \) and \( I(f) \). The value of \( I(f) \), which we denote by \( T_{2,2} \), is an improved approximation given by

\[
T_{2,2} = T_{2,1} + \frac{T_{2,1} - T_{1,1}}{3}.
\]
It follows from the representation of the error in the Composite Trapezoidal Rule that \( I(f) = T_{2,2} + O(h^4) \).

Suppose that we compute another approximation \( T_{3,1} \) using the Composite Trapezoidal Rule with 4 subintervals. Then, as before, we can use Richardson Extrapolation with \( T_{2,1} \) and \( T_{3,1} \) to obtain a new approximation \( T_{3,2} \) that is fourth-order accurate. Now, however, we have two approximations, \( T_{2,2} \) and \( T_{3,2} \), that satisfy
\[
T_{2,2} = I(f) + \tilde{K}_2 h^4 + O(h^6)
\]
\[
T_{3,2} = I(f) + \tilde{K}_2 (h/2)^4 + O(h^6)
\]
for some constant \( \tilde{K}_2 \). It follows that we can apply Richardson Extrapolation to these approximations to obtain a new approximation \( T_{3,3} \) that is sixth-order accurate. We can continue this process to obtain as high an order of accuracy as we wish. We now describe the entire algorithm.

**Algorithm (Romberg Integration)** Given a positive integer \( J \), an interval \([a, b]\) and a function \( f(x) \), the following algorithm computes an approximation to \( I(f) = \int_a^b f(x) \, dx \) that is accurate to order \( 2J \).

\( h = b - a \)

for \( j = 1, 2, \ldots, J \) do

\[ T_{j,1} = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{2^{j-1}-1} f(a + jh) + f(b) \right] \] (Composite Trapezoidal Rule)

for \( k = 2, 3, \ldots, j \) do

\[ T_{j,k} = T_{j,k-1} + \frac{T_{j,k-1} - T_{j-1,k-1}}{4^k - 1} \] (Richardson Extrapolation)

end

\( h = h/2 \)

end

It should be noted that in a practical implementation, \( T_{j,1} \) can be computed more efficiently by using \( T_{j-1,1} \), because \( T_{j-1,1} \) already includes more than half of the function values used to compute \( T_{j,1} \), and they are weighted correctly relative to one another. It follows that for \( j > 1 \), if we split the summation in the algorithm into two summations containing odd- and even-numbered terms, respectively, we obtain

\[
T_{j,1} = \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{2^{j-2}} f(a + (2j - 1)h) + 2 \sum_{j=1}^{2^{j-2}-1} f(a + 2jh) + f(b) \right]
\]

\[
= \frac{h}{2} \left[ f(a) + 2 \sum_{j=1}^{2^{j-2}-1} f(a + 2jh) + f(b) \right] + \frac{h}{2} \left[ 2 \sum_{j=1}^{2^{j-2}-1} f(a + (2j - 1)h) \right]
\]

\[
= \frac{1}{2} T_{j-1,1} + h \sum_{j=1}^{2^{j-2}} f(a + (2j - 1)h).
\]
Example We will use Romberg integration to obtain a sixth-order accurate approximation to
\[ \int_{0}^{1} e^{-x^2} \, dx, \]
an integral that cannot be computed using the Fundamental Theorem of Calculus. We begin by
using the Trapezoidal Rule, or, equivalently, the Composite Trapezoidal Rule
\[ \int_{a}^{b} f(x) \, dx \approx \frac{h}{2} \left[ f(a) + \sum_{j=1}^{n-1} f(x_j) + f(b) \right], \quad h = \frac{b-a}{n}, \quad x_j = a + jh, \]
with \( n = 1 \) subintervals. Since \( h = (b-a)/n = (1-0)/1 = 1 \), we have
\[ R_{1,1} = \frac{1}{2} [f(0) + f(1)] = 0.68393972058572, \]
which has an absolute error of \( 6.3 \times 10^{-2} \).

If we bisect the interval \([0,1]\) into two subintervals of equal width, and approximate the area
under \( e^{-x^2} \) using two trapezoids, then we are applying the Composite Trapezoidal Rule with \( n = 2 \)
and \( h = (1-0)/2 = 1/2 \), which yields
\[ R_{2,1} = \frac{0.5}{2} [f(0) + 2f(0.5) + f(1)] = 0.73137025182856, \]
which has an absolute error of \( 1.5 \times 10^{-2} \). As expected, the error is reduced by a factor of 4 when
the step size is halved, since the error in the Composite Trapezoidal Rule is of \( O(h^2) \).

Now, we can use Richardson Extrapolation to obtain a more accurate approximation,
\[ R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{3} = 0.74718042890951, \]
which has an absolute error of \( 3.6 \times 10^{-4} \). Because the error in the Composite Trapezoidal Rule satisfies
\[ \int_{a}^{b} f(x) \, dx = \frac{h}{2} \left[ f(a) + \sum_{j=1}^{n-1} f(x_j) + f(b) \right] + K_1 h^2 + K_2 h^4 + K_3 h^6 + O(h^8), \]
where the constants \( K_1, K_2 \) and \( K_3 \) depend on the derivatives of \( f(x) \) on \([a,b]\) and are independent
of \( h \), we can conclude that \( R_{2,1} \) has fourth-order accuracy.

We can obtain a second approximation of fourth-order accuracy by using the Composite Trape-
zoidal Rule with \( n = 4 \) to obtain a third approximation of second-order accuracy. We set \( h = (1-0)/4 = 1/4 \), and then compute
\[ R_{3,1} = \frac{0.25}{2} [f(0) + 2[f(0.25) + f(0.5) + f(0.75)] + f(1)] = 0.74298409780038, \]
which has an absolute error of $3.8 \times 10^{-3}$. Now, we can apply Richardson Extrapolation to $R_{2,1}$ and $R_{3,1}$ to obtain

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{3} = 0.74685537979099,$$

which has an absolute error of $3.1 \times 10^{-5}$. This significant decrease in error from $R_{2,2}$ is to be expected, since both $R_{2,2}$ and $R_{3,2}$ have fourth-order accuracy, and $R_{3,2}$ is computed using half the step size of $R_{2,2}$.

It follows from the error term in the Composite Trapezoidal Rule, and the formula for Richardson Extrapolation, that

$$R_{2,2} = \int_0^1 e^{-x^2} \, dx + \tilde{K}_2 h^4 + O(h^6), \quad R_{2,2} = \int_0^1 e^{-x^2} \, dx + \tilde{K}_2 \left(\frac{h}{2}\right)^4 + O(h^6).$$

Therefore, we can use Richardson Extrapolation with these two approximations to obtain a new approximation

$$R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{2,2}}{2^4 - 1} = 0.74683370984975,$$

which has an absolute error of $9.6 \times 10^{-6}$. Because $R_{3,3}$ is a linear combination of $R_{3,2}$ and $R_{2,2}$ in which the terms of order $h^4$ cancel, we can conclude that $R_{3,3}$ is of sixth-order accuracy. □