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Lecture 11 Notes

These notes correspond to Section 6.2 in the text.

Pivoting Strategies

We have seen that during Gaussian elimination, it is necessary to interchange rows of the augmented matrix whenever the diagonal element of the column currently being processed, known as the *pivot element*, is equal to zero.

However, if we examine the main step in Gaussian elimination,

$$a_{ik}^{(j+1)} = a_{ik}^{(j)} - m_{ij}a_{jk}^{(j)},$$

we can see that any roundoff error in the computation of $a_{jk}^{(j)}$ is amplified by m_{ij} . Therefore, it is helpful if it can be ensured that the multipliers are small. This can be accomplished by performing row interchanges, or *pivoting*, even when it is not absolutely necessary to do so for elimination to proceed.

Partial Pivoting

One approach is called *partial pivoting*. When eliminating elements in column j , we seek the largest element in column j , on or below the main diagonal, and then interchanging that element's row with row j . That is, we find an integer p , $j \leq p \leq n$, such that

$$|a_{pj}| = \max_{j \leq i \leq n} |a_{ij}|.$$

Then, we interchange rows p and j .

In view of the definition of the multiplier,

$$m_{ij} = \frac{a_{ij}^{(j)}}{a_{jj}^{(j)}},$$

it follows that $|m_{ij}| \leq 1$ for $j = 1, \dots, n-1$ and $i = j+1, \dots, n$. Furthermore, while pivoting in this manner requires $O(n^2)$ comparisons to determine the appropriate row interchanges, that extra expense is negligible compared to the overall cost of Gaussian elimination, and therefore is outweighed by the potential reduction in roundoff error.

Scaled Partial Pivoting

While partial pivoting helps to control the propagation of roundoff error, loss of significant digits can still result if, in the abovementioned main step of Gaussian elimination, $m_{ij}a_{jk}^{(j)}$ is much larger in magnitude than $a_{ij}^{(j)}$. Even though m_{ij} not large, this can still occur if $a_{jk}^{(j)}$ is particularly large.

To work around this, we modify partial pivoting by first, implicitly, scaling all of the rows so that the largest element in each row has a magnitude of 1. Then, we apply partial pivoting to the resulting matrix. Specifically, we first compute *scale factors*

$$s_i = \max_{1 \leq j \leq n} |a_{ij}|,$$

and then, when eliminating elements in column j , we interchange row j with row p , where

$$\frac{|a_{pj}|}{s_p} = \max_{j \leq i \leq n} \frac{|a_{ij}|}{s_i}.$$

This technique is called *scaled partial pivoting*. It can produce multipliers that are larger than 1 in magnitude, but it is still more effective than partial pivoting at containing roundoff error. Furthermore, only $O(n^2)$ comparisons are required to compute the scale factors, so it does not add significant overhead to Gaussian elimination.

Complete Pivoting

Ideally, the scaling factors from scaled partial pivoting should be updated as the matrix is modified during Gaussian elimination. However, this would result in a total of $O(n^3)$ comparisons, which is much more expensive than computing the scale factors once at the beginning. If this much effort is to be expended to determine which row interchanges should be performed, it is preferable to instead use the most robust pivoting technique available, *complete pivoting*.

Complete pivoting entails finding integers p and q such that

$$|a_{pq}| = \max_{j \leq i \leq n, j \leq q \leq n} |a_{ij}|,$$

and then using both row *and column* interchanges to move a_{pq} into the pivot position in row j and column j . It has been proven that this is an effective strategy for ensuring that Gaussian elimination is *backward stable*, meaning it does not cause the entries of the matrix to grow exponentially as they are updated by elementary row operations, which is undesirable because it can cause undue amplification of roundoff error.