These notes correspond to Section 5.7 in the text.

**Variable Step-Size Multistep Methods**

Previously, we learned how to use two one-step methods in order to estimate the local truncation error, and then use this estimate to select the step size $h$ needed to achieve a given level of accuracy. Now, we consider the same problem of selecting the proper step size with multistep methods.

Unlike one-step methods, where we use two methods of different order of accuracy, we select an explicit and implicit multistep method that have the same order. That is, we use an $s$-step explicit Adams-Bashforth method whose local truncation error is $O(h^s)$, and an $(s-1)$-step implicit Adams-Moulton method whose local truncation error is also $O(h^s)$.

In order to estimate the local truncation error of the explicit method, we need to know the theoretical local truncation error term for both methods. To obtain this, we integrate the error in the polynomial interpolation used to derive the method from $t_n$ to $t_{n+1}$. For the explicit $s$-step method, this yields

$$\tau_{n+1}(h) = \frac{1}{h} \int_{t_n}^{t_{n+1}} \frac{f(s)(\xi; y(\xi))}{s!} (t - t_n)(t - t_{n-1}) \cdots (t - t_{n-s+1}) dt.$$  

Using the substitution $u = (t_{n+1} - t)/h$, and the Weighted Mean Value Theorem for Integrals, yields

$$\tau_{n+1}(h) = \frac{1}{h} \frac{f(s)(\xi; y(\xi))}{s!} h^{s+1} (-1)^s \int_0^1 (u-1)(u-2) \cdots (u-s) du.$$  

Evaluating the integral yields the constant in the error term. We also use the fact that $y' = f(t, y)$ to replace $f^{(s)}(\xi; y(\xi))$ with $y^{(s+1)}(\xi)$. Obtaining the local truncation error for an implicit, Adams-Moulton method can be accomplished in the same way, except that $t_{n+1}$ is also used as an interpolation point.

We now assume that our explicit $s$-step method has local truncation error of the form

$$\tilde{\tau}_{n+1}(h) = C_1 y^{(s+1)}(\xi_n) h^s,$$

for $t_n \leq \xi_n \leq t_{n+1}$, and that the $(s-1)$-step implicit method has local truncation error

$$\tau_{n+1}(h) = C_2 y^{(s+1)}(\eta_n) h^s,$$

for $t_n \leq \eta_n \leq t_{n+1}$. 

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where \( t_n \leq \eta_n \leq t_{n+1} \). If we assume that \( y_0, y_1, \ldots, y_n \) are exact, it follows that

\[
\tilde{y}_{n+1}(h) = \frac{y(t_{n+1}) - \tilde{y}_{n+1}}{h}, \quad \tau_{n+1}(h) = \frac{y(t_{n+1}) - y_{n+1}}{h},
\]

where \( \tilde{y}_{n+1} \) and \( y_{n+1} \) are the values computed using the explicit and implicit method, respectively, in the framework of a predictor-corrector method.

We assume that \( y^{(s+1)}(\xi_n) \approx y^{(s+1)}(\eta_n) \), and then subtract the local truncation errors, to obtain

\[
\frac{y_{n+1} - \tilde{y}_{n+1}}{h} \approx (C_1 - C_2)h^s y^{(s+1)}(\xi_n),
\]

which yields

\[
y^{(s+1)}(\eta_n) \approx \frac{1}{(C_1 - C_2)h^{s+1}}(y_{n+1} - \tilde{y}_{n+1}).
\]

We substitute this expression for \( y^{(s+1)}(\eta_n) \) into the local truncation error \( \tau_{n+1}(h) \) to obtain

\[
|\tau_{n+1}(h)| = |\frac{y(t_{n+1}) - y_{n+1}}{h}| \approx |C_2|h^s \frac{1}{(C_1 - C_2)h^{s+1}}|y_{n+1} - \tilde{y}_{n+1}| = \left| \frac{C_2}{h(C_1 - C_2)} \right| |y_{n+1} - \tilde{y}_{n+1}|.
\]

If we multiply the step size \( h \) by \( q \), this multiples \( \tau_{n+1}(h) \) by \( q^s \). It follows that in order to ensure that \( |\tau_{n+1}(h)| < \epsilon \) for some tolerance \( \epsilon \), we must have

\[
q^s \left| \frac{C_2}{h(C_1 - C_2)} \right| |y_{n+1} - \tilde{y}_{n+1}| < \epsilon,
\]

or

\[
q < \left( \left| \frac{C_1 - C_2}{C_2} \right| \frac{eh}{|y_{n+1} - \tilde{y}_{n+1}|} \right)^{1/s}.
\]

**Example** Let \( s = 4 \). Then, our predictor-corrector method uses the four-step explicit Adams-Bashforth method

\[
y_{n+1} = y_n + \frac{h}{24}[55f(t_n, y_n) - 59f(t_{n-1}, y_{n-1}) + 37f(t_{n-2}, y_{n-2}) - 9f(t_{n-3}, y_{n-3})]
\]

and the three-step implicit Adams-Moulton method

\[
y_{n+1} = y_n + \frac{h}{24}[9f(t_{n+1}, y_{n+1}) + 19f(t_n, y_n) - 5f(t_{n-1}, y_{n-1}) + f(t_{n-2}, y_{n-2})].
\]

We first use the explicit method to compute \( \tilde{y}_{n+1} \), and then use \( f(t_{n+1}, \tilde{y}_{n+1}) \) in the implicit method to compute \( y_{n+1} \), as in any Adams-Moulton predictor-corrector method.

The local truncation error for the 4-step explicit Adams-Bashforth method is

\[
\tilde{\tau}_{n+1}(h) = y^{(5)}(\xi_n) \frac{h^4}{4!} \int_0^1 (u-1)(u-2)(u-3)(u-4) \, du = \frac{251}{720} y^{(5)}(\xi_n) h^4,
\]
and the local truncation error for the 3-step implicit Adams-Moulton method is

\[ \tau_{n+1}(h) = -\frac{19}{720} y^{(5)}(\eta_n) h^4. \]

That is, \( C_1 = \frac{251}{720} \) and \( C_2 = -\frac{19}{720} \). It follows that we should choose \( q \) so that

\[ q < \left( \frac{270}{19} \frac{\epsilon h}{|y_{n+1} - \tilde{y}_{n+1}|} \right)^{1/4}. \]

\[ \square \]

Using adaptive step size control is more expensive for multistep methods than for one-step methods, because once the step size \( h \) is changed, it is necessary to use a one-step method such as fourth-order Runge-Kutta to compute \((s - 1)\) new starting values, at equally spaced times with spacing equal to the new step size \( h \). For this reason, the time step is only changed if a significant change is called for, and even then, in practice the step size is kept between lower and upper bounds, as it is in adaptive step size control for one-step methods.