These notes correspond to Section 1.1 in the text.

Examples of Dynamical Systems

This course is devoted to the study of systems of ordinary differential equations (ODEs), in terms of analytical and numerical solution techniques, and also acquiring insight into the qualitative behavior of solutions. We begin our study of a general system of \( n \) ODEs of the form

\[
y' = F(t, y),
\]

where \( y(t) \colon \mathbb{R} \to \mathbb{R}^n \) and \( F : \mathbb{R}^{n+1} \to \mathbb{R}^n \), with some examples that illustrate very different types of qualitative behavior. The behavior that we can observe has led to the following common synonym.

**Definition 1 (Dynamical System)** A dynamical system is a system of ordinary differential equations.

**Example 1 (Circular Flow)** We begin with the simple dynamical system

\[
\begin{align*}
x' &= y \\
y' &= -x.
\end{align*}
\]

By differentiating the first equation, we obtain \( x'' = -x \), which has the general solution

\[
x(t) = A \cos t + B \sin t
\]

where \( A \) and \( B \) are constants. It follows from \( x' = y \) that \( y(t) = B \cos t - A \sin t \). If we express the point \((A, B)\) in polar coordinates \( A = r \cos \theta \) and \( B = r \sin \theta \), then, using trigonometric identities, we conclude that this dynamical system has the solution

\[
x(t) = r \cos(\theta - t), \quad y(t) = r \sin(\theta - t).
\]

That is, the graph of any solution is a circle of radius \( r \) centered at the origin. Because the graph is a closed curve, we say that this circle is the *orbit* of the solution. \( \square \)

**Example 2 (A Row of Stagnation Points)** Next, we consider the dynamical system

\[
\begin{align*}
x' &= \sinh y \\
y' &= -\sin x
\end{align*}
\]

Unlike the preceding example, an explicit formula for the solution cannot be found analytically. However, we can still understand the behavior of the solution. The following concept is helpful.

**Definition 2 (Streamline)** A streamline, or solution curve or integral curve, of a dynamical system \( y' = F(t, y) \) is a curve \( r(t) \) defined for \( t \) in an interval \( I \), such that \( r'(t) = F(t, r(t)) \) for \( t \in I \). That is, at each point on the curve, its tangent vector is given by \( F(t, r(t)) \).
Streamlines of this dynamical system can be computed using numerical methods, which will be discussed in an upcoming lecture.

Upon plotting the streamlines of this system, it can be seen that at certain points, there are no visible streamlines; that is, the flow isn’t “going anywhere”. That is, the velocity of the streamline is zero. This leads to the following concept.

**Definition 3 (Fixed Point)** A fixed point, also known as a stagnation point, stationary point, or equilibrium point of a dynamical system $y' = F(y)$ is a point at which $F(y) = 0$.

It can readily be determined that the dynamical system in this example has fixed points at $(x, y) = (k\pi, 0)$ for each integer $k$. Furthermore, as $|y|$ increases, the streamlines approach horizontal lines, with flow from left to right for $y > 0$ and flow from right to left for $y < 0$. This can be seen by examining the velocity vectors of the streamlines, $(x', y') = (\sinh y, \sin x)$, as $|y|$ increases.

**Example 3 (Karmen Vortex Sheet)** Consider the system

$$
\begin{align*}
x' &= \frac{\cos x \sinh y - \sinh c}{M(x, y)}, \\
y' &= -\frac{\sin x \cosh y}{M(x, y)},
\end{align*}
$$

where $c = 0.8828$ and

$$M(x, y) = \frac{[\cosh(y + c) + \cos x][\cosh(y - c) - \cos x]}{2 \cosh c}.
$$

It can be seen that $y' = 0$ when $x = k\pi$, for any integer $k$, but for such $x$, the numerator in $x'$ is nonzero except when $y = \pm c$. However, when this is the case, $M(x, y) = 0$, so the derivatives are actually undefined. That is, this system has no fixed points.

That said, these points at which $M(x, y) = 0$ are still noteworthy. For $x = k\pi$ for even $k$, and $y = c$, or when $x = k\pi$ for odd $k$ and $y = -c$, the velocity of the streamline passing through these points is infinite, as can be determined using l’Hospital’s Rule. Such a point is referred to as a vortex.

**Example 4 (Heat Flow in a Cube)** Next, we examine a three-dimensional dynamical system

$$
\begin{align*}
x' &= b \sin(2\pi x) \cos(2\pi y) \sinh(2\sqrt{2}\pi z) \\
y' &= b \cos(2\pi x) \sin(2\pi y) \sinh(2\sqrt{2}\pi z), \\
z' &= -1 - \sqrt{2} b \cos(2\pi x) \cos(2\pi y) \cosh(2\sqrt{2}\pi z)
\end{align*}
$$

that models heat flow in the unit cube $[0, 1]^3$. Let $S(x, y, z)$ denote the temperature throughout the cube. Then, because heat energy flows from hot to cold, the heat flux vector field is $-\nabla S$, the direction of steepest descent of $S$. That is, $-\nabla S$ indicates the direction of streamlines, and therefore we have

$$x' = -S_x, \quad y' = -S_y, \quad z' = -S_z.$$

Comparing these equations to the original dynamical system (8) yields

$$S(x, y, z) = z + [\cos(2\pi x) \cos(2\pi y) \sinh(2\sqrt{2}\pi z)]/\sinh(2\sqrt{2}\pi).$$

Note that this dynamical system has no fixed points in the interior of the cube.
Example 5 (The Two-Body Problem)

\[
\begin{align*}
    m_1 r_1'' &= \frac{G m_1 m_2 (r_2 - r_1)}{r_1^3} \\
    m_2 r_2'' &= \frac{G m_1 m_2 (r_1 - r_2)}{r_2^3}
\end{align*}
\]

(9) \hspace{1cm} (10)

Unlike the previous examples, this system consists of second-order equations rather than first-order. However, any higher-order system can be rewritten as a first-order system.

**Definition 4 (Phase Space)** Let \( F : D \rightarrow \mathbb{R}^n \), where \( D \subseteq \mathbb{R}^n \). The domain \( D \) of \( F \) is called the phase space of the dynamical system \( y' = F(y) \).

In this case, the phase space consists of \( U \times \mathbb{R}^6 \), where \( U \equiv \{(r_1, r_2) | r_1 \neq r_2 \} \). The additional 6 variables are used for the conversion of the original second-order system into a first-order system, which adds 6 equations, one for each dependent variable in the second-order system. \( \square \)

Example 6 (Pendulum/Ball in a Hoop) The motion of a pendulum, or really, a weight on a string, can be modeled using the second-order equation

\[
\theta'' = -k \sin \theta,
\]

where \( \theta \) is the angle that the pendulum makes with the downward axis. As in the previous example, this equation can easily be converted into a system of two first-order equations,

\[
\begin{align*}
    \theta' &= v, \\
    v' &= -k \sin \theta.
\end{align*}
\]

The phase space of this first-order system is all of \( \mathbb{R}^2 \).

This equation is too simplified to accurately model the motion of the pendulum, because, for example, from the position \( \theta = 3\pi/4 \), the weight would first drop straight down until the slack in the string is taken up. Therefore, we impose constrained motion and instead use this equation to model the motion of a ball in a hoop.

It can be seen from the first-order system that the points \((\theta, v) = (n\pi, 0)\), where \( n \) is an integer, are fixed points. However, these fixed points are associated with very different behavior of the solution, depending on whether \( n \) is odd or even. If \( n \) is even, the ball is at the bottom, so if the ball is released, it remains stationary. Therefore, we say that these points are stable fixed points. If \( n \) is odd, then the ball is at the top of the hoop, and if it is released, it will quickly move far away from the top, so the fixed point is unstable.

A phase portrait, or a graph of streamlines in phase space, illustrates the very different behavior of solutions to this system. The phase portrait for this system is shown in Figure 1. First, it can be seen that the stable fixed points are the centers of closed orbits. Second, once \( |v| \) is sufficiently large, the streamlines are no longer orbital; instead they are wavy curves that oscillate around a constant value of \( v \). Physically, this means that once the ball has enough initial velocity, it can cycle through the hoop indefinitely. Finally, there are two streamlines that pass through the unstable fixed points, and clearly define a separation between the orbital and wavy streamlines. These streamlines are called separatrices for the system.

The phase portrait in Figure 1 was generated in MATLAB, using the following function:

```matlab
function phaseportrait(xp,yp,a,b,c,d,h,xname,yname)
x=a:h:b;
```

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Figure 1: Phase portrait for dynamical system $\theta' = v$, $v' = -k \sin \theta$ with $k = 1$

\begin{verbatim}
y=c:h:d;
[x2,y2]=meshgrid(x,y);
u2=xp(x2,y2);
v2=yp(x2,y2);
z2=u2./sqrt(u2.^2+v2.^2);
w2=v2./sqrt(u2.^2+v2.^2);
quiver(x2,y2,z2,w2);
xlabel(xname)
ylabel(yname)
axis tight
\end{verbatim}

Then, the function was used as follows:

\begin{verbatim}
>> xp=inline('y','x','y');
>> yp=inline('-sin(x)','x','y');
>> phaseportrait(xp,yp,-10,10,-4,4,0.2,'\theta','v');
\end{verbatim}

\hspace{1cm} □

**Example 7 (Perturbed Pendulum/Ball in a Hoop)** In this example, we modify the previous motion of the ball in a hoop by allowing the hoop to rotate through an angle $a$ with frequency $b$. Assuming the hoop itself has radius 1, we obtain the following modified ODE

$$\theta'' = -g \sin \theta + \frac{a^2 b^2}{2} \sin^2(bt) \sin(2\theta)$$

which can be written as a first-order dynamical system

\begin{align*}
\theta' &= v \tag{11} \\
v' &= -g \sin(\theta) + \frac{a^2 b^2}{2} \sin^2(bt) \sin(2\theta). \tag{12}
\end{align*}

This system differs from all of the preceding examples in that $v'$ depends directly on the independent variable $t$. It follows that this system cannot be written in the form $y' = F(y)$ for some function $F$; instead, we must use the more general form $y' = F(t, y)$. This leads to the following definition.
Definition 5 (Autonomous System) A dynamical system \( \dot{y} = F(t, y) \) is said to be autonomous, or time-independent, if \( F \) is independent of \( t \); that is, an autonomous system can be written in the form \( \dot{y} = F(y) \). When this is not the case, the system is said to be non-autonomous, or time-dependent.

It is far easier to study the behavior of autonomous systems, as we can look for fixed points, etc. but fortunately, any non-autonomous system can be rewritten as an autonomous system by introducing an extra variable:

\[
\begin{align*}
\tau' &= 1 \\
\theta' &= v \\
v' &= -g \sin(\theta) + \frac{a^2 b^2}{2} \sin^2(b \tau) \sin(2 \theta).
\end{align*}
\]

If \( \tilde{\alpha}(t) = (t, \theta(t), v(t)) \) is a solution curve of the autonomous system (13), then a solution curve of the original first-order system (11), (12) can easily be obtained by projecting \( \tilde{\alpha} \) onto the \( \theta-v \) plane. It follows that a solution curve of a non-autonomous system can cross itself, whereas a solution curve for an autonomous system cannot.

For some dynamical systems, even if is not possible to obtain an explicit formula for solution curves, it may be possible to describe them implicitly. Consider a system of the form

\[
x' = f(x, y), \quad y' = g(x, y).
\]

Multiplying the first equation by \( y' \) and the second equation by \( x' \) leads to the equation

\[
f(x, y) dy = g(x, y) dx.
\]

This is a first-order ODE that can be solved if, for example, it is separable or exact, or if an integrating factor can be found that makes it separable or exact.

If an implicit solution of the form \( F(x, y) = 0 \) can be found, then the function \( F \) is said to be a first integral of the dynamical system. The solution curves are the level curves of \( F \); that is, \( F(x, y) = k \) describes a solution curve for any constant \( k \) in the range of \( F \). It is possible to find a first integral for the system in Example 2, so at least an implicit analytical solution can be found, if not an explicit one.

Exercises

Section 1.1: Exercise 4