This assignment is due on Tuesday, April 18. In this homework and later ones, please note the following:

- **Bold face** lower case letters, *e.g.* \(x\), usually refer to vectors.
- Upper case letters, *e.g.* \(A\), usually refer to matrices.
- Lower case letters, *e.g.* \(a\), usually refer to scalars.

1. Suppose we wish to solve 
\[Ax = b.\]
Show that if there exists a diagonal matrix \(D\) such that 
\[B = DAD^{-1}\]
is symmetric and positive definite, then the SOR method converges for the original problem.

*Hint:* Use the fact that SOR converges for any symmetric positive definite matrix.

2. We want to consider the symmetric successive overrelaxation (SSOR) method. The method is described as follows:
\[
\begin{align*}
y^{(k+1)}_i &= x^{(k)}_i + \frac{\omega}{a_{ii}} \left( b_i - a_{ii}x^{(k)}_i - \sum_{j=1}^{i-1} a_{ij}y^{(k+1)}_j - \sum_{j=i+1}^{n} a_{ij}x^{(k)}_j \right), \quad i = 1, 2, \ldots, n \\
x^{(k+1)}_i &= y^{(k+1)}_i + \frac{\omega}{a_{ii}} \left( b_i - a_{ii}y^{(k+1)}_i - \sum_{j=1}^{i-1} a_{ij}y^{(k+1)}_j - \sum_{j=i+1}^{n} a_{ij}x^{(k+1)}_j \right), \quad i = n, n-1, \ldots, 1 
\end{align*}
\]
or in matrix notation,
\[
\begin{align*}
y^{(k+1)} &= x^{(k)} + \omega D^{-1} \left( b - Ly^{(k+1)} - Ux^{(k)} - Dx^{(k)} \right) \\
x^{(k+1)} &= y^{(k+1)} + \omega D^{-1} \left( b - Ly^{(k+1)} - Ux^{(k+1)} - Dy^{(k+1)} \right)
\end{align*}
\]
We assume \(A\) is symmetric and positive definite.

(a) Construct the operator \(S_\omega = M_\omega^{-1}N_\omega\) such that \(x^{(k+1)} = S_\omega x^{(k)} + M_\omega^{-1}b.\)
(b) Show that SSOR is equivalent to writing the system as
\[M_\omega x^{(k+1)} = N_\omega x^{(k)} + b.\]

Find \(M_\omega.\)
(c) Apply SSOR to solving Laplace’s equation on an \((N+1) \times (N+1)\) grid with \(N = 23\). The problem is to be solved for \(0 < x < 1\), and \(0 < y < 1\), with the following boundary conditions:

\[
\begin{align*}
ü(0, y) &= 0, & u(1, y) &= y, \\
u(x, 0) &= 0, & u(x, 1) &= x^2
\end{align*}
\]

Choose a random starting vector on \((0, 1)\), and iterate until the relative error is less than \(10^{-5}\). Solve the problem for five values of \(\omega\) and find an approximate “optimal” value \(\hat{\omega}\). (You may want to use \texttt{polyfit} to fit a polynomial to the obtained number of iterations as function of \(\omega\) and find its minimum.)

3. Let \(A\) be an \(n \times n\) symmetric matrix that is singular. Consider the iteration

\[
M x^{(k+1)} = N x^{(k)} + b.
\]

(a) Show that at least one eigenvalue of \(B = M^{-1}N\) equals 1.

(b) Describe how you could modify a convergent algorithm to obtain a solution to a singular system. \textit{Hints:} For general \(b\), \(Ax = b\) does not have a solution; in such cases, your algorithm should instead solve \(Ax = \tilde{b}\), where \(\tilde{b}\) is in the column space of \(A\). If the iteration is not converging for the given \(b\), then how \textit{is} it behaving? Because \(A\) is symmetric, how does the range of \(A\) relate to its null space? Use the fact that for any vector \(x \in \mathbb{R}^n\), and any subspace \(S \subseteq \mathbb{R}^n\), there exists a unique decomposition \(x = x_S + x_S^\perp\), where \(x_S \in S\) and \(x_S^\perp \in S^\perp\), the orthogonal complement of \(S\). The vector \(x_S\) is the \textit{orthogonal projection} of \(x\) onto \(S\) (see the notes).

Use the above iteration to show how the difference between two consecutive iterates, \(x^{(k+1)} - x^{(k)}\), relates to the difference between the previous two iterates, \(x^{(k+1)} - x^{(k-1)}\). For a convergent iteration, this difference will converge to zero, but what does it converge to if \(A\) is singular, based on your work on part (a)? And how can this be used to “fix” the iteration?

(c) Using Gauss-Seidel, compute a solution to the problem

\[
Ax = b
\]

where \(A\) is an \(n \times n\) matrix of the form

\[
A = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & -1 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & 0 & \ldots \\
& & & & & \\
0 & \ldots & 0 & -1 & 2 & -1 \\
-1 & 0 & \ldots & 0 & -1 & 2
\end{bmatrix},
\]

with \(n\) even, and \(b\) is any pre–specified vector. Describe your modification to obtain a solution.