MAT 415 DIFFERENTIAL EQUATIONS II
HOMEWORK 2

(1) Show that

(a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$

(b) $\sin x = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x)$.

Hint: Using the generating function $g(x,t)$ write e^{ix} and e^{-ix} as series with Bessel functions for their coefficients.

(2) Show, by direct differentiation, that

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s}$$

satisfies the two recurrence relations.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x),$$
$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

and Bessel's differential equation

$$x^2 J_n''(x) + x J_n'(x) + (x^2 - n^2) J_n(x) = 0.$$

(3) A particle with mass m is contained in a right circular cylinder (pillbox) of radius R and height H. The particle is described by a wave function satisfying the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\rho,\varphi,z) = E\psi(\rho,\varphi,z)$$

and the condition that the wave function go to zero over the surface of the pillbox. Find zero point energy, i.e. the lowest permitted energy.

(4) The amplitude $U(\rho, \varphi, t)$ of a vibrating circular membrane of radisu a satisfies the wave equation

$$\nabla^2 U - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0.$$

Here v is the phase velocity of the wave fixed by the elastic constants and whatever damping is imposed.

(a) Show that a solution is

$$U(\rho, \varphi, t) = J_m(k\rho)(a_1 e^{im\varphi} + a_2 e^{-im\varphi})(b_1 e^{i\omega t} + b_2 e^{-i\omega t}).$$

(b) From the Dirichlet boundary condition $J_m(ka) = 0$, find the allowable values of the wave length λ ($k = \frac{2\pi}{\lambda}$).