MAT 415 DIFFERENTIAL EQUATIONS II
HOMEWORK 4

(1) Use the transformation $R(kr) = \frac{Z(kr)}{(kr)^{1/2}}$ to show that the equation

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0$$

can be written as the Bessel's equation

$$r^2 \frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - \left(n + \frac{1}{2}\right)^2\right] Z = 0.$$

(2) Show that if

$$n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x),$$

it automatically equals

$$(-1)^{n+1}\sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).$$

(3) (a) Derive the recurrence relations

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x),$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x),$$

satisfied by the spherical Bessel functions $j_n(x)$ and $n_n(x)$.

(b) Show, from these two recurrence relations, that the spherical Bessel function $f_n(x)$ satisfies the differential equation

$$x^2 f_n''(x) + 2x f_n'(x) + [x^2 - n(n+1)] f_n(x) = 0.$$

(4) A hollow sphere of radius a (Helmholtz resonator) contains standing sound waves. Find the minimum frequency of oscillation in terms of the radius a and the velocity of sound v. The sound waves satisfy the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

and the boundary condition

$$\frac{\partial\psi}{\partial r}|_{r=a} = 0.$$

This is a Neumann boundary condition.