STABILITY ANALYSIS OF KRYLOV SUBSPACE SPECTRAL METHODS FOR THE 1-D WAVE EQUATION IN INHOMOGENEOUS MEDIA

by

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ABSTRACT

Krylov subspace spectral (KSS) methods are high-order accurate, explicit time-stepping methods for partial differential equations (PDEs) that also possess the stability characteristic of implicit methods. Unlike other time-stepping approaches, KSS methods compute each Fourier coefficient of the solution from an individualized approximation of the solution operator of the PDE. As a result, KSS methods scale effectively to higher spatial resolution. This thesis will present a stability analysis of a first-order KSS method applied to the wave equation in inhomogeneous media.

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

The problem to be analyzed is the wave equation with a variable coefficients [2]

$$u_{tt} = (p(x)u_x)_x + q(x)u, \quad 0 < x < 2\pi, \quad t > 0 \tag{1.1}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < 2\pi \tag{1.2}$$

and periodic boundary conditions

$$u(0,t) = u(2\pi,t), \quad t > 0. \tag{1.3}$$

We seek a numerically stable method to ensure the computed solution remains bounded. If the method is unconditionally stable, then regardless of the choice of Δt, the computed solution will remain bounded.

Analytical methods, such as separation of variables, are not practical to use for solving this problem since the coefficients are not constant and would result in a spatial ODE that we do not know how to solve analytically. Numerical methods for solving this type of PDE such as finite difference methods require a CFL constraint [4] such that the time-step must be proportional to the grid spacing. Therefore, at high resolution it would require a very small time step. Standard time-stepping methods such as Euler's, Runge-Kutta, multi-step methods would not be useful due to their lack of scalability. As the number of grid points increases, a smaller time step would be needed due to a CFL condition for some methods or an ill-conditioned system for implicit methods. Increasing the number of grid points dramatically increases the computation time. Therefore, a more practical numerical method for solving this kind of variable-coefficient PDE is needed.

Krylov Subspace Spectral (KSS) methods are high-order accurate, explicit time-stepping methods for partial differential equations (PDEs) that also possess the stability characteristic of implicit methods [8]. Unlike other time-stepping approaches, KSS methods compute each Fourier coefficient of the solution from an individualized approximation of the solution operator of the PDE. As a result, KSS methods scale effectively to higher spatial resolution.

This thesis project builds on ideas from stability results of KSS methods. A 1-node KSS method applied to the heat equation with a constant leading coefficient was proven to be unconditionally stable [7, 6], as well as when applied to the wave equation with a constant leading coefficient [5]. Furthermore, a 1-node KSS method applied to the heat equation with a variable leading coefficient is unconditionally stable [8]. In this thesis, we analyzed of a stability of a KSS method applied to the wave equation with a variable coefficients under certain assumptions.

The outline of the thesis is as follows. Chapter 2 provides an overview of KSS methods. In Chapter 3 we perform a stability analysis of a first-order KSS method applied to the wave equation with a variable leading coefficient and include a proof that the method is unstable. Also, in Chapter 3 we prove that under certain assumptions, the method is unconditionally stable. Conclusions and ideas for future work are given in Chapter 4.

Chapter 2
BACKGROUND

2.1 Derivation of KSS Methods

We begin with the derivation of KSS methods which is easier to explain for the heat equation.

2.1.1 From PDEs to Bilinear Forms

For simplicity, consider a linear parabolic PDE of the form

$$u_t + Lu = 0$$

where L is a second-order, self-adjoint differential operator. Then the solution can be expressed as Fourier series where each Fourier coefficient

$$\hat{u}(\omega, t) = \langle e^{i\omega x}, e^{-Lt}u(x,0)\rangle$$

is an inner product of Fourier basis functions and the solution operator applied to the initial data. Spatial discretization results in a bilinear form involving a matrix function of the form

$$\mathbf{u}^H f(A)\mathbf{v}$$

where $f(A) = e^{-At}$.

2.1.2 Elements of Functions of Matrices

Golub and Meurant described in their 1994 paper “Matrices, Moments and Quadrature" [3] a method for computing bilinear forms such as

$$\mathbf{u}^T f(A)\mathbf{v},$$

where $\mathbf{u}$ and $\mathbf{v}$ are N-vectors, A is an $N \times N$ symmetric positive definite matrix, and f is a smooth function. Since the matrix A is symmetric positive definite it has real, positive eigenvalues

$$b = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N = a > 0,$$

and orthogonal corresponding eigenvectors $\mathbf{q}_j$, $j = 1, \ldots, N$. Then using its spectral decomposition, the matrix A can be rewritten as

$$A = \sum_{j=1}^{N} \lambda_j \mathbf{q}_j \mathbf{q}_j^T.$$

2.1.3 Using the Spectral Decomposition

Then computing the matrix function $f(A)$ only requires evaluating f at each eigenvalue λ_j and can be rewritten as

$$f(A) = \sum_{j=1}^{N} f(\lambda_j)\mathbf{q}_j\mathbf{q}_j^T.$$

Then for the bilinear form, we take the same summation and multiply by $\mathbf{u}^T$ on the left and $\mathbf{v}$ on the right. So altogether, the bilinear form $\mathbf{u}^T f(A)\mathbf{v}$ can expressed as

$$\mathbf{u}^T f(A)\mathbf{v} = \sum_{j=1}^{N} f(\lambda_j)\mathbf{u}^T\mathbf{q}_j\mathbf{q}_j^T\mathbf{v},$$

but computing this summation would be impractical for large values of N because computing all of the eigenvalues and eigenvectors would require using the QR algorithm which would take order $O(N^3)$ FLOPS (floating point operations).

2.1.4 From Bilinear Forms to Integrals

We let $\alpha(\lambda)$ be a step function defined in terms of the coefficients of $\mathbf{u}$ and $\mathbf{v}$ in the basis of eigenvectors u_j and v_j,

$$\alpha(\lambda) = \begin{cases} 0, & \text{if } \lambda < a \\ \sum_{j=i}^{N} u_j v_j, & \text{if } \lambda_i \leq \lambda < \lambda_{i-1} \\ \sum_{j=1}^{N} u_j v_j, & \text{if } b \leq \lambda \end{cases}, \quad u_j = \mathbf{u}^T\mathbf{q}_j, \quad v_j = \mathbf{q}_j^T\mathbf{v}$$

where $a = \lambda_N$ is the smallest eigenvalue and $b = \lambda_1$ is the largest eigenvalue. Then the bilinear form $\mathbf{u}^T f(A)\mathbf{v}$ can be viewed as a Riemann-Stieljes integral over the spectral domain

$$\begin{aligned} \mathbf{u}^T f(A)\mathbf{v} &= \sum_{j=1}^{N} f(\lambda_j)\mathbf{u}^T\mathbf{q}_j\mathbf{q}_j^T\mathbf{v} \\ &= \sum_{j=1}^{N} f(\lambda_j)u_j v_j \\ &= \int_a^b f(\lambda)\, d\alpha(\lambda). \end{aligned}$$

2.1.5 Approximation via Gauss Quadrature

This integral can then be approximated using a Gauss quadrature rule resulting in

$$\sum_{j=1}^{K} w_j f(\lambda_j) + R[f],$$

where the nodes λ_j, $j = 1, \ldots, K$, and the weights w_j, $j = 1, \ldots, K$ can be obtained from the Lanczos algorithm applied to A with initial vectors $\mathbf{u}$ and $\mathbf{v}$. This results in a tridiagonal matrix T_K where the eigenvalues are the nodes, and the first components of the eigenvalues squared are the weights. When the vectors $\mathbf{u}$ and $\mathbf{v}$ are equal, $\alpha(\lambda)$ is positive and increasing, but if $\mathbf{u} \neq \mathbf{v}$ this may not necessarily be the case which can lead to numerically unstable quadrature rule.

2.1.6 Block Gaussian Quadrature

As an alternative, we use a block approach in which we compute

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^T f(A) \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$$

which guarantees the weights will be positive. Computing this block quadratic form results in the 2×2 matrix integral

$$\begin{aligned} \int_a^b f(\lambda)\, d\alpha(\lambda) &= \begin{bmatrix} \mathbf{u}^T f(A)\mathbf{u} & \mathbf{u}^T f(A)\mathbf{v} \\ \mathbf{v}^T f(A)\mathbf{u} & \mathbf{v}^T f(A)\mathbf{v} \end{bmatrix} \\ &= \sum_{j=1}^{2K} f(\lambda_j)\mathbf{u}_j\mathbf{u}_j^T + \text{error} \end{aligned}$$

where λ_j is a scalar, and $\mathbf{u}_j$ is a 2-vector. We can approximate this integral using a quadrature rule with twice as many nodes as the non-block case where the outer product $(\mathbf{u}_j\mathbf{u}_j^T)$ is the weight.

2.1.7 Computation of Block Gaussian Quadrature Rules

The nodes and weights are obtained by generalizing the Lanczos algorithm to the block case. Applying the block Lanczos algorithm to A with initial block $[\mathbf{u}, \mathbf{v}]$ results in the $2K \times 2K$ block tridiagonal matrix

$$\mathcal{T}_K = \begin{bmatrix} M_1 & B_1^T & & & \\ B_1 & M_2 & B_2^T & & \\ & \ddots & \ddots & \ddots & \\ & & B_{K-2} & M_{K-1} & B_{K-1}^T \\ & & & B_{K-1} & M_K \end{bmatrix}.$$

We then define the quadrature rule for $\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}^T f(A) \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ as

$$\int_a^b f(\lambda)\, d\alpha(\lambda) \approx \sum_{j=1}^{2K} f(\lambda_j)\mathbf{u}_j\mathbf{u}_j^T = [f(\mathcal{T}_K)]_{1:2,1:2}$$

where λ_j an eigenvalue of $\mathcal{T}_K$, and $\mathbf{u}_j$ contains the first two elements of the normalized eigenvector.

2.2 KSS Methods for the Wave Equation

2.2.1 The Wave Equation

Now, we apply these ideas to the second-order wave equation

$$u_{tt} + Lu = 0 \text{ on } (0,2\pi)\times(0,\infty)$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 < x < 2\pi$$

with periodic boundary conditions

$$u(0,t) = u(2\pi,t), \qquad t > 0.$$

The spatial differential operator L is defined by

$$Lu = -(p(x)u_x)_x + q(x)u$$

where we assume $p(x) > 0$ and $q(x) \geq 0$ to guarantee L is self-adjoint and positive-definite.

2.2.2 Application to the Wave Equation

A spectral representation of the operator L allows us to obtain a representation of the solution operator, the propagator. We introduce

$$R_1(t) = L^{-1/2}\sin(t\sqrt{L}), \quad R_0(t) = \cos(t\sqrt{L})$$

. Then the solution can be written as

$$\begin{bmatrix} u(x,t) \\ u_t(x,t) \end{bmatrix} = \begin{bmatrix} R_0(t) & R_1(t) \\ -LR_1(t) & R_0(t) \end{bmatrix} \begin{bmatrix} u(x,0) \\ u_t(x,0) \end{bmatrix}.$$

The entries of this matrix, as functions of L, indicate which functions are the integrands in the Riemann-Stieltjes integrals used to compute the Fourier components of the solution.

2.2.3 Krylov Subspace Spectral Methods

For each wave number $\omega = -N/2+1,\ldots,N/2$, we define

$$R_0 = \begin{bmatrix} \hat{\mathbf{e}}_\omega & \mathbf{u}^n \end{bmatrix}, \quad \tilde{R}_0 = \begin{bmatrix} \hat{\mathbf{e}}_\omega & \mathbf{u}_t^n \end{bmatrix},$$

and then compute the QR factorizations

$$R_0 = X_1 B_0, \quad \tilde{R}_0 = \tilde{X}_1 \tilde{B}_0.$$

Block Lanczos iteration yields $\mathcal{T}_K$ and $\tilde{\mathcal{T}}_K$ from X_1 and $\tilde{X}_1$. Then the solution and its time derivative are approximated by

$$[\hat{\mathbf{u}}^{n+1}]_\omega = \left[B_0^H \cos[\mathcal{T}_K^{1/2}\Delta t]_{1:2,1:2} B_0\right]_{12} + \left[\tilde{B}_0^H (\tilde{\mathcal{T}}_K^{-1/2} \sin[\tilde{\mathcal{T}}_K^{1/2}\Delta t])_{1:2,1:2} \tilde{B}_0\right]_{12},$$

$$[\hat{\mathbf{u}}_t^{n+1}]_\omega = -\left[B_0^H (\mathcal{T}_K^{1/2} \sin[\mathcal{T}_K^{1/2}\Delta t])_{1:2,1:2} B_0\right]_{12} + \left[\tilde{B}_0^H \cos[\tilde{\mathcal{T}}_K^{1/2}\Delta t]_{1:2,1:2} \tilde{B}_0\right]_{12}.$$

2.2.4 Consistency and Stability

Let $u(x,\Delta t)$ be the exact solution, and let $\tilde{u}(x,\Delta t)$ be the approximate solution. If K quadrature nodes are used, then [6]

$$|\langle \hat{\mathbf{e}}_\omega, u(\cdot,\Delta t) - \tilde{u}(\cdot,\Delta t)\rangle| = O(\Delta t^{4K}),$$

$$|\langle \hat{\mathbf{e}}_\omega, u_t(\cdot,\Delta t) - \tilde{u}_t(\cdot,\Delta t)\rangle| = O(\Delta t^{4K-1}).$$

Furthermore, KSS methods represent a best-of-both worlds compromise between the computational efficiency of explicit methods and the stability of implicit methods. This combination is achieved through a componentwise approach in which each Fourier coefficient of the solution is computed from an individualized approximation of the solution operator of the PDE. As a result, KSS methods scale effectively to higher spatial resolution.

2.3 Stability Results for KSS

This thesis investigates the stability of a 2-node block KSS method for a class of wave equations using ideas from previous stability results for KSS methods applied to other problems. The following has been proven about the stability of KSS methods:

- Heat equation: $u_t = pu_{xx} + q(x)u$, where p is constant, $q(x)$ bandlimited, a 1-node KSS method is unconditionally stable [7]

- Wave equation: $u_{tt} = pu_{xx} + q(x)u$, where p is constant, $q(x)$ bandlimited, a 1-node KSS method is unconditionally stable [5]

- Reaction-diffusion system of the form $\mathbf{v}_t = L\mathbf{v}$: a first-order KSS method is unconditionally stable [8]

- Wave equation: $u_{tt} = pu_{xx} + q(x)u$, where p is constant, $q(x)$ bandlimited, a 2-node block KSS method is unconditionally stable [5]

- Heat equation: $u_t = (p(x)u_x)_x + q(x)u$ with variable $p(x)$ and where $p(x)$ is bandlimited, a 2-node block KSS method is unconditionally stable [8].

Chapter 3

STABILITY ANALYSIS

The PDE we will be analyzing stability for is the wave equation with variable coefficients,

$$u_{tt} = (p(x)u_x)_x + q(x)u, \tag{3.1}$$

under the assumptions that $\hat{p}(\omega) = 0$ and $\hat{q}(\omega) = 0$ for $|\omega| > \omega_{\max}$.

3.1 Norm of the Solution Operator

We define the exact solution operator as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} R_0(t) & R_1(t) \\ -LR_1(t) & R_0(t) \end{bmatrix} \tag{3.2}$$

where $R_1(t) = L^{-1/2}\sin(t\sqrt{L})$ and $R_0(t) = \cos(t\sqrt{L})$. Then we let

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12} & \tilde{P}_{22} \end{bmatrix}$$

where each $\tilde{P}_{ij}$ is the approximation of P_{ij} by KSS. To establish stability, we need an upper bound for a norm of the solution operator $\tilde{P}$ that maps $[u^n, u_t^n]$ to $[u^{n+1}, u_t^{n+1}]$. We use the C-norm defined by

$$\| (\mathbf{u}, \mathbf{v}) \|_C^2 = \mathbf{u}^T C\mathbf{u} + \|\mathbf{v}\|_2^2$$

where C is an $N \times N$ matrix that represents the constant-coefficient differential operator $-\bar{p}u_{xx} + \bar{q}$, where u and v are N-vectors. The notation $\bar{f}$ denotes the mean of a function $f(x)$ on $[0, 2\pi]$. We choose to bound the C-norm of the solution operator for convenience, because the operator C has a very simple expression in Fourier space due to the constant coefficients which simplifies the analysis. Now, we want to express $\|\tilde{P}\|_C$ as the 2-norm of some matrix instead, since that will be easier to bound. We let $\|\tilde{P}\|_C^2 = \sup_{\mathbf{w}=(\mathbf{u},\mathbf{v})\neq\mathbf{0}} \dfrac{\|\tilde{P}\mathbf{w}\|_C^2}{\|\mathbf{w}\|_C^2}$. Then

$$\|\tilde{P}\|_C^2 = \sup \frac{\tilde{\mathbf{u}}^T C\tilde{\mathbf{u}} + \|\tilde{\mathbf{v}}_2\|^2}{\mathbf{u}^T C\mathbf{u} + \|\mathbf{v}_2\|^2}, \tag{3.3}$$

where $\begin{bmatrix}\tilde{\mathbf{u}}\\ \tilde{\mathbf{v}}\end{bmatrix} = \tilde{P}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = \tilde{P}\mathbf{w}$. In matrix form, we have

$$\begin{aligned}\|\tilde{P}\|_C^2 &= \sup \frac{\mathbf{w}^T\tilde{P}^T\tilde{C}\tilde{P}\mathbf{w}}{\mathbf{w}^T\tilde{C}\mathbf{w}} \\ &= \sup \frac{\mathbf{w}^T\tilde{P}^T\tilde{C}\tilde{P}\mathbf{w}}{(\tilde{C}^{1/2}\mathbf{w})^T(\tilde{C}^{1/2}\mathbf{w})}\end{aligned}$$

where $\tilde{C} = \begin{bmatrix}C & 0\\ 0 & I\end{bmatrix}$. Let $\mathbf{z} = \tilde{C}^{1/2}\mathbf{w}$. Then

$$\begin{aligned}\|\tilde{P}\|_C^2 &= \sup \frac{\mathbf{w}^T\tilde{P}^T\tilde{C}\tilde{P}\mathbf{w}}{(\tilde{C}^{1/2}\mathbf{w})^T(\tilde{C}^{1/2}\mathbf{w})} \\ &= \sup \frac{\mathbf{z}^T(\tilde{C}^{1/2}\tilde{P}\tilde{C}^{-1/2})^T(\tilde{C}^{1/2}\tilde{P}\tilde{C}^{-1/2})\mathbf{z}}{\mathbf{z}^T\mathbf{z}}.\end{aligned}$$

Therefore,

$$\|\tilde{P}\|_C = \|\tilde{C}^{1/2}\tilde{P}\tilde{C}^{-1/2}\|_2 = \|B\|_2 = \sqrt{\rho\left(B^HB\right)} \leq \sqrt{\|G\|_\infty}$$

where

$$G = B^HB = \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix} = \begin{bmatrix}C^{-1/2}\tilde{P}_{11}^HC^{1/2} & C^{-1/2}\tilde{P}_{21}^H\\ \tilde{P}_{12}^HC^{1/2} & \tilde{P}_{22}^H\end{bmatrix}\begin{bmatrix}C^{1/2}\tilde{P}_{11}C^{-1/2} & C^{1/2}\tilde{P}_{12}\\ \tilde{P}_{21}C^{-1/2} & \tilde{P}_{22}\end{bmatrix}. \quad (3.4)$$

3.2 Stability

We obtain an expression for the approximate solution operator $\tilde{P}$ so that we can bound its C-norm. We have $\mathbf{u}^{n+1} = P_{11}(\Delta t)\mathbf{u}^n + P_{12}(\Delta t)\mathbf{u}_t^n$ and $\mathbf{u}_t^{n+1} = P_{21}(\Delta t)\mathbf{u}^n + P_{22}(\Delta t)\mathbf{u}_t^n$. This yields

$$\begin{aligned}\|(\mathbf{u}^{n+1},\mathbf{u}_t^{n+1})\|_C^2 &= (\mathbf{u}^{n+1})^TC(\mathbf{u}^{n+1}) + (\mathbf{u}_t^{n+1})^T(\mathbf{u}_t^{n+1}) \\ &= (\tilde{P}_{11}(\Delta t)\mathbf{u}^n + \tilde{P}_{12}(\Delta t)\mathbf{u}_t^n)^TC(\tilde{P}_{11}(\Delta t)\mathbf{u}^n + \tilde{P}_{12}(\Delta t)\mathbf{u}_t^n) + \\ &\quad (\tilde{P}_{21}(\Delta t)\mathbf{u}^n + P_{22}(\Delta t)\mathbf{u}_t^n)^T(P_{21}(\Delta t)\mathbf{u}^n + \tilde{P}_{22}(\Delta t)\mathbf{u}_t^n) \\ &= (\tilde{P}_{11}(\Delta t)\mathbf{u}^n)^TC(\tilde{P}_{11}(\Delta t)\mathbf{u}^n) + (\tilde{P}_{21}(\Delta t)\mathbf{u}^n)^T(P_{21}(\Delta t)\mathbf{u}^n) + \\ &\quad (\tilde{P}_{11}(\Delta t)\mathbf{u}^n)^TC(\tilde{P}_{12}(\Delta t)\mathbf{u}_t^n) + (\tilde{P}_{21}(\Delta t)\mathbf{u}^n)^T(\tilde{P}_{22}(\Delta t)\mathbf{u}_t^n) + \\ &\quad (\tilde{P}_{12}(\Delta t)\mathbf{u}_t^n)^TC(P_{11}(\Delta t)\mathbf{u}^n) + (P_{22}(\Delta t)\mathbf{u}_t^n)^T(\tilde{P}_{21}(\Delta t)\mathbf{u}^n) + \\ &\quad (P_{12}(\Delta t)\mathbf{u}_t^n)^TC(P_{12}(\Delta t)\mathbf{u}_t^n) + (P_{22}(\Delta t)\mathbf{u}_t^n)^T(P_{22}(\Delta t)\mathbf{u}_t^n).\end{aligned}$$

We define $\mathbf{z}_{11} = \tilde{\mathbf{P}}_{11}(\Delta t)\mathbf{u}^n$, $\mathbf{z}_{21} = \tilde{\mathbf{P}}_{21}(\Delta t)\mathbf{u}^n$, $\mathbf{z}_{12} = \tilde{\mathbf{P}}_{12}(\Delta t)\mathbf{u}_t^n$, and $\mathbf{z}_{22} = \tilde{\mathbf{P}}_{22}(\Delta t)\mathbf{u}_t^n$. Using the fact that KSS methods compute individualized approximations of each Fourier coefficient, by working with the Fourier coefficients of the approximate solution, we obtain

$$\begin{aligned}
\|(\mathbf{u}^{n+1},\mathbf{u}_t^{n+1})\|_C^2 &= \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{11}(\omega)(\bar{p}\omega^2+\bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{21}(\omega) + \\
&\quad \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{12}(\omega)(\bar{p}\omega^2+\bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{22}(\omega) + \\
&\quad \sum_\omega \overline{\hat{\mathbf{z}}_{12}(\omega)}\hat{\mathbf{z}}_{11}(\omega)(\bar{p}\omega^2+\bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{22}(\omega)}\hat{\mathbf{z}}_{21}(\omega) + \\
&\quad \sum_\omega \overline{\hat{\mathbf{z}}_{12}(\omega)}\hat{\mathbf{z}}_{12}(\omega)(\bar{p}\omega^2+\bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{22}(\omega)}\hat{\mathbf{z}}_{22}(\omega) \\
&= [\mathbf{u}^n]^T G_{11}\mathbf{u}^n + [\mathbf{u}^n]^T G_{12}\mathbf{u}_t^n + [\mathbf{u}_t^n]^T G_{21}\mathbf{u}^n + [\mathbf{u}_t^n]^T G_{22}\mathbf{u}_t^n \qquad (3.5)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\mathbf{z}}_{11}(\omega) &= P_{11}(l_{2,\omega})(\hat{e}_\omega^H\mathbf{u}^n) + M_{11,\omega}\hat{e}_\omega^H(A-l_{2,\omega}I)\mathbf{u}^n \\
&= P_{11}(l_{2,\omega})\hat{u}(\omega) - i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}[\hat{p}(\omega-k)i(k)\hat{u}(k)] + \\
&\quad M_{11,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}\hat{q}(\omega-k)\hat{u}(k) \\
\hat{\mathbf{z}}_{21}(\omega) &= P_{21}(l_{2,\omega})(\hat{e}_\omega^H\mathbf{u}^n) + M_{21,\omega}\hat{e}_\omega^H(A-l_{2,\omega}I)\mathbf{u}^n \\
&= P_{21}(l_{2,\omega})\hat{u}(\omega) - i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}[\hat{p}(\omega-k)i(k)\hat{u}(k)] + \\
&\quad M_{21,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}\hat{q}(\omega-k)\hat{u}(k) \\
\hat{\mathbf{z}}_{12}(\omega) &= P_{12}(l_{2,\omega})\hat{\mathbf{u}}_t(\omega) - i\omega M_{12,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}[\hat{p}(\omega-k)i(k)\hat{\mathbf{u}}_t(k)] + \\
&\quad M_{12,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}\hat{q}(\omega-k)\hat{\mathbf{u}}_t(k) \\
\hat{\mathbf{z}}_{22}(\omega) &= P_{22}(l_{2,\omega})\hat{\mathbf{u}}_t(\omega) - i\omega M_{22,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}[\hat{p}(\omega-k)i(k)\hat{\mathbf{u}}_t(k)] + \\
&\quad M_{22,\omega}\frac{1}{\sqrt{N}}\sum_{k\neq\omega}\hat{q}(\omega-k)\hat{\mathbf{u}}_t(k),
\end{aligned}$$

A is a matrix that discretizes the spatial differential operator of the PDE, $l_{2,\omega} = \bar{p}\omega^2 + \bar{q}$ and $l_{1,\omega} = \bar{q}$ are approximations of the Gauss quadrature nodes produced by block Lanczos [1], to improve efficiency without losing accuracy, and

$$
\begin{aligned}
\hat{e}_\omega^H (A - l_{2,\omega} I)\mathbf{u}_t^n &= \hat{e}_\omega^H A\mathbf{u}_t^n - l_{2,\omega}(\hat{e}_\omega^H \mathbf{u}_t^n) \\
&= \hat{e}_\omega^H[-DPD+Q]\mathbf{u}_t^n - l_{2,\omega}(\hat{e}_\omega^H \mathbf{u}_t^n) \\
&= -i\omega \hat{e}_\omega^H PD\mathbf{u}_t^n + \hat{e}_\omega^H Q\mathbf{u}_t^n - l_{2,\omega}(\hat{e}_\omega^H \mathbf{u}_t^n) \\
&= -i\omega \bar{p}\hat{e}_\omega^H D\mathbf{u}_t^n - i\omega \hat{e}_\omega^H \tilde{p} D\mathbf{u}_t^n + \bar{q}\hat{e}_\omega^H \mathbf{u}_t^n + \hat{e}_\omega^H \tilde{Q}\mathbf{u}_t^n - l_{2,\omega}(\hat{e}_\omega^H \mathbf{u}_t^n) \\
&= (\bar{p}\omega^2 + \bar{q})(\hat{e}_\omega^H \mathbf{u}_t^n) - i\omega \hat{e}_\omega^H \tilde{p} D\mathbf{u}_t^n + \hat{e}_\omega^H \tilde{Q}\mathbf{u}_t^n - l_{2,\omega}(\hat{e}_\omega^H \mathbf{u}_t^n) \\
&= -i\omega \hat{e}_\omega^H \tilde{p} D\mathbf{u}_t^n + \hat{e}_\omega^H \tilde{Q}\mathbf{u}_t^n
\end{aligned}
$$

where $\tilde{p} = p - \bar{p}$, $\tilde{q} = q - \bar{q}$, D is a differentiation matrix, and P, Q, and $\tilde{Q}$ are diagonal matrices with the values of the coefficients on the diagonal.

To find a bound for the overall approximate solution operator $\tilde{P}$, we proceed by bounding G block by block.

Lemma 3.2.1. *Assume $\hat{p}(\omega) = 0$ and $\hat{q}(\omega) = 0$ for $|\omega| \geq \omega_{\max}$. Then the matrix G_{11} defined in (3.4) satisfies*

$$
\begin{aligned}
\|G_{11}\|_\infty \leq\; & 1 + C_{11,p}\|\tilde{p}\|_\infty \Delta t^2 N^2 + C_{11,q}\|\tilde{q}\|_\infty \Delta t^2 + C_{11,p^2}\|\tilde{p}\|_\infty^2 \Delta t^2 N^2 + \\
& C_{11,pq}\|\tilde{p}\|_\infty \|\tilde{q}\|_\infty \Delta t^2 + C_{11,q^2}\|\tilde{q}\|_\infty^2 \Delta t^2
\end{aligned} \tag{3.6}
$$

where each constant is independent of N and Δt.

Proof. From (3.5) we have

$$
[\mathbf{u}^n]^T G_{11}\mathbf{u}^n = \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{11}(\omega)(\bar{p}\omega^2 + \bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{21}(\omega).
$$

To evaluate this expression, we work out the first term and second term separately, then add them together. We have

$$\begin{aligned}
\overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{11}(\omega) &= \Bigg(P_{11}(l_{2,\omega})\hat{u}(\omega) - i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)] + \\
&\quad M_{11,\omega}\frac{1}{\sqrt{N}}\sum_k \hat{q}(\omega-k)\hat{u}(k)\Bigg)\Bigg(P_{11}(l_{2,\omega})\hat{u}(-\omega) - i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\times \\
&\quad \sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + M_{11,\omega}\frac{1}{\sqrt{N}}\sum_j \hat{q}(j-\omega)\hat{u}(-j)\Bigg) \\
&= (P_{11}(l_{2,\omega}))^2\hat{u}(\omega)\hat{u}(-\omega) - P_{11}(l_{2,\omega})\hat{u}(\omega)i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\times \\
&\quad \sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + P_{11}(l_{2,\omega})\hat{u}(\omega)M_{11,\omega}\frac{1}{\sqrt{N}}\sum_j[\hat{q}(j-\omega)\hat{u}(-j)] - \\
&\quad i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]P_{11}(l_{2,\omega})\hat{u}(-\omega) - \\
&\quad \omega^2 M_{11,\omega}^2\frac{1}{N}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] - \\
&\quad i\omega M_{11,\omega}^2\frac{1}{N}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]\sum_j[\hat{q}(j-\omega)\hat{u}(-j)] + \\
&\quad M_{11,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]P_{11}(l_{2,\omega})\hat{u}(-\omega) - \\
&\quad i\omega M_{11,\omega}^2\frac{1}{N}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + \\
&\quad M_{11,\omega}^2\frac{1}{N}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]\sum_j[\hat{q}(j-\omega)\hat{u}(-j)].
\end{aligned}$$

Let $s_1 = \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{11}(\omega)(\overline{p}\omega^2 + \overline{q})$. Then

$$
\begin{aligned}
s \;=\; & \sum_{\omega}(P_{11}(l_{2,\omega}))^2\hat{u}(\omega)\hat{u}(-\omega)(\overline{p}\omega^2+\overline{q})- \\
& \sum_{j}\sum_{\omega}\frac{1}{\sqrt{N}}\hat{u}(-j)i(j)P_{11}(l_{2,\omega})i\omega M_{11,\omega}\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\hat{u}(\omega)+ \\
& \sum_{j}\sum_{\omega}\frac{1}{\sqrt{N}}\hat{u}(-j)P_{11}(l_{2,\omega})M_{11,\omega}\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\hat{u}(\omega)- \\
& \sum_{k}\sum_{\omega}\frac{1}{\sqrt{N}}\hat{u}(k)i(k)P_{11}(l_{2,\omega})i\omega M_{11,\omega}\hat{p}(\omega-k)(\overline{p}\omega^2+\overline{q})\hat{u}(-\omega)- \\
& \sum_{j}\sum_{k}\frac{1}{N}\hat{u}(k)i(k)\hat{u}(-j)i(j)\sum_{\omega}\omega^2M_{11,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})- \\
& \sum_{j}\sum_{k}\frac{1}{N}\hat{u}(k)i(k)\hat{u}(-j)\sum_{\omega}i\omega M_{11,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})+ \\
& \sum_{k}\sum_{\omega}\frac{1}{\sqrt{N}}\hat{u}(k)P_{11}(l_{2,\omega})M_{11,\omega}\hat{q}(\omega-k)(\overline{p}\omega^2+\overline{q})\hat{u}(-\omega)- \\
& \sum_{j}\sum_{k}\frac{1}{N}\hat{u}(k)\hat{u}(-j)i(j)\sum_{\omega}i\omega M_{11,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})+ \\
& \sum_{j}\sum_{k}\frac{1}{N}\hat{u}(k)\hat{u}(-j)\sum_{\omega}M_{11,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q}) \\
= \; & \sum_{j}\hat{u}(j)\hat{u}(-j)\left[(P_{11}(l_{2,j}))^2(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{11}(l_{2,k})ikM_{11,k}\hat{p}(j-k)(\overline{p}k^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,k})M_{11,k}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(k)P_{11}(l_{2,j})ijM_{11,j}\hat{p}(j-k)(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_{\omega}\omega^2M_{11,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)\sum_{\omega}i\omega M_{11,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{11,j}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(j)\sum_{\omega}i\omega M_{11,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_{\omega}M_{11,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{21}(\omega) &= \Bigg(P_{21}(l_{2,\omega})\hat{u}(\omega) - i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)] + \\
&\quad M_{21,\omega}\frac{1}{\sqrt{N}}\sum_k \hat{q}(\omega-k)\hat{u}(k)\Bigg)\Bigg(P_{21}(l_{2,\omega})\hat{u}(-\omega) - i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\times \\
&\quad \sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j \hat{q}(j-\omega)\hat{u}(-j)\Bigg) \\
&= (P_{21}(l_{2,\omega}))^2\hat{u}(\omega)\hat{u}(-\omega) - P_{21}(l_{2,\omega})\hat{u}(\omega)i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\times \\
&\quad \sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + P_{21}(l_{2,\omega})\hat{u}(\omega)M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j[\hat{q}(j-\omega)\hat{u}(-j)] - \\
&\quad i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]P_{21}(l_{2,\omega})\hat{u}(-\omega) - \\
&\quad \omega^2 M_{21,\omega}^2\frac{1}{N}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] - \\
&\quad i\omega M_{21,\omega}^2\frac{1}{N}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}(k)]\sum_j[\hat{q}(j-\omega)\hat{u}(-j)] + \\
&\quad M_{21,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]P_{21}(l_{2,\omega})\hat{u}(-\omega) - \\
&\quad i\omega M_{21,\omega}^2\frac{1}{N}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + \\
&\quad M_{21,\omega}^2\frac{1}{N}\sum_k[\hat{q}(\omega-k)\hat{u}(k)]\sum_j[\hat{q}(j-\omega)\hat{u}(-j)]
\end{aligned}
$$

which yields

$$
\begin{aligned}
\sum_{\omega}\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{21}(\omega) \quad = \quad & \sum_{\omega}(P_{21}(l_{2,\omega}))^2\hat{u}(\omega)\hat{u}(-\omega)- \\
& \sum_j\sum_{\omega}\hat{u}(-j)\hat{u}(\omega)i(j)P_{21}(l_{2,\omega})i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\hat{p}(j-\omega)+ \\
& \sum_j\sum_{\omega}\hat{u}(-j)\hat{u}(\omega)\frac{1}{\sqrt{N}}P_{21}(l_{2,\omega})M_{21,\omega}\hat{q}(j-\omega)- \\
& \sum_k\sum_{\omega}\hat{u}(k)\hat{u}(-\omega)i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\hat{p}(\omega-k)i(k)P_{21}(l_{2,\omega})- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\frac{1}{N}i(k)i(j)\sum_{\omega}\omega^2M_{21,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)i\frac{1}{N}i(k)\sum_{\omega}\omega M_{21,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)+ \\
& \sum_k\sum_{\omega}\hat{u}(k)\hat{u}(-\omega)\frac{1}{\sqrt{N}}P_{21}(l_{2,\omega})M_{21,\omega}\hat{q}(\omega-k)- \\
& \sum_j\sum_k\hat{u}(k)i(j)\hat{u}(-j)i\frac{1}{N}\sum_{\omega}\omega M_{21,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)+ \\
& \sum_j\sum_k\hat{u}(k)\hat{u}(-j)\frac{1}{N}\sum_{\omega}M_{21,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega) \\
= \quad & \sum_j\hat{u}(j)\hat{u}(-j)\left[(P_{21}(l_{2,j}))^2\right]- \qquad (3.7) \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{21}(l_{2,k})ikM_{21,k}\hat{p}(j-k)\right]+ \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,k})M_{21,k}\hat{q}(j-k)\right]- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[ijM_{21,j}\frac{1}{\sqrt{N}}\hat{p}(j-k)i(k)P_{21}(l_{2,j})\right]- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_{\omega}\omega^2M_{21,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)\right]- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[i\frac{1}{N}i(k)\sum_{\omega}\omega M_{21,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)\right]+ \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{21,j}\hat{q}(j-k)\right]- \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[i(j)i\frac{1}{N}\sum_{\omega}\omega M_{21,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)\right]+ \\
& \sum_j\sum_k\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_{\omega}M_{21,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)\right] \qquad (3.8)
\end{aligned}
$$

Putting the two equations (3.7) and (3.8) together, we obtain

$$
\begin{aligned}
& \sum_{\omega}\overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{11}(\omega)(\bar{p}\omega^2+\bar{q})+\sum_{\omega}\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{21}(\omega) \\
= & \sum_{j}\hat{u}(j)\hat{u}(-j)\left[(P_{11}(l_{2,j}))^2(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{11}(l_{2,k})ikM_{11,k}\hat{p}(j-k)(\overline{p}k^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,k})M_{11,k}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(k)P_{11}(l_{2,j})ijM_{11,j}\hat{p}(j-k)(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_{\omega}\omega^2M_{11,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)\sum_{\omega}i\omega M_{11,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{11,j}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(j)\sum_{\omega}i\omega M_{11,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_{\omega}M_{11,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right]+ \\
& \sum_{j}\hat{u}(j)\hat{u}(-j)\left[(P_{21}(l_{2,j}))^2\right]-\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{21}(l_{2,k})ikM_{21,k}\hat{p}(j-k)\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,k})M_{21,k}\hat{q}(j-k)\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[ijM_{21,j}\frac{1}{\sqrt{N}}\hat{p}(j-k)i(k)P_{21}(l_{2,j})\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_{\omega}\omega^2M_{21,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[i\frac{1}{N}i(k)\sum_{\omega}\omega M_{21,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{21,j}\hat{q}(j-k)\right]- \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[i(j)i\frac{1}{N}\sum_{\omega}\omega M_{21,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)\right]+ \\
& \sum_{j}\sum_{k}\hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_{\omega}M_{21,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \sum_j \hat{u}(j)\hat{u}(-j)\left[(P_{11}(l_{2,j}))^2(\overline{p}j^2+\overline{q})\right] + \sum_j \hat{u}(j)\hat{u}(-j)\left[(P_{21}(l_{2,j}))^2\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{11}(l_{2,k})ikM_{11,k}\hat{p}(j-k)(\overline{p}k^2+\overline{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,k})M_{11,k}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(j)P_{21}(l_{2,k})ikM_{21,k}\hat{p}(j-k)\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,k})M_{21,k}\hat{q}(j-k)\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}i(k)P_{11}(l_{2,j})ijM_{11,j}\hat{p}(j-k)(\overline{p}j^2+\overline{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{11,j}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[ijM_{21,j}\frac{1}{\sqrt{N}}\hat{p}(j-k)i(k)P_{21}(l_{2,j})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{21,j}\hat{q}(j-k)\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_\omega \omega^2 M_{11,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)\sum_\omega i\omega M_{11,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(j)\sum_\omega i\omega M_{11,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_\omega M_{11,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}i(k)i(j)\sum_\omega \omega^2 M_{21,\omega}^2\hat{p}(\omega-k)\hat{p}(j-\omega)\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[i\frac{1}{N}i(k)\sum_\omega \omega M_{21,\omega}^2\hat{p}(\omega-k)\hat{q}(j-\omega)\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[i(j)i\frac{1}{N}\sum_\omega \omega M_{21,\omega}^2\hat{q}(\omega-k)\hat{p}(j-\omega)\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[\frac{1}{N}\sum_\omega M_{21,\omega}^2\hat{q}(\omega-k)\hat{q}(j-\omega)\right] \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}(k)\left[A+B+C+D\right]_{jk}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{jj} &= \left(P_{11}(l_{2,j})\right)^2 (\overline{p}j^2+\overline{q}) + \left(P_{21}(l_{2,j})\right)^2 \\
&= \left(P_{11}(l_{2,j})\right)^2 (\overline{p}j^2+\overline{q}) + \left(P_{21}(l_{2,j})\right)^2 \\
&= \cos^2\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right)\left(\overline{p}j^2+\overline{q}\right) + \left(-\left(\overline{p}j^2+\overline{q}\right)^{1/2}\sin\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right)\right)^2 \\
&= \cos^2\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right)(\overline{p}j^2+\overline{q}) + (\overline{p}j^2+\overline{q})\sin^2\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right) \\
&= \overline{p}j^2+\overline{q},
\end{aligned}
$$

$$
\begin{aligned}
B_{jk} = {} & \frac{1}{\sqrt{N}}i(j)P_{11}(l_{2,k})ikM_{11,k}\hat{p}(j-k)(\overline{p}k^2+\overline{q}) + \frac{1}{\sqrt{N}}P_{11}(l_{2,k})M_{11,k}\hat{q}(j-k)(\overline{p}k^2+\overline{q}) + \\
& \frac{1}{\sqrt{N}}i(j)P_{21}(l_{2,k})ikM_{21,k}\hat{p}(j-k) + \frac{1}{\sqrt{N}}P_{21}(l_{2,k})M_{21,k}\hat{q}(j-k),
\end{aligned}
$$

$$
\begin{aligned}
C_{jk} = {} & \frac{1}{\sqrt{N}}i(k)P_{11}(l_{2,j})ijM_{11,j}\hat{p}(j-k)(\overline{p}j^2+\overline{q}) + \frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{11,j}\hat{q}(j-k)(\overline{p}j^2+\overline{q}) + \\
& ijM_{21,j}\frac{1}{\sqrt{N}}\hat{p}(j-k)i(k)P_{21}(l_{2,j}) + \frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{21,j}\hat{q}(j-k),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{jk} &= \frac{1}{N} i(k)i(j)\sum_{\omega} \omega^2 M_{11,\omega}^2 \hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \\
&\quad \frac{1}{N} i(k)\sum_{\omega} i\omega M_{11,\omega}^2 \hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \\
&\quad \frac{1}{N} i(j)\sum_{\omega} i\omega M_{11,\omega}^2 \hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \\
&\quad \frac{1}{N} \sum_{\omega} M_{11,\omega}^2 \hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \\
&\quad \frac{1}{N} i(k)i(j)\sum_{\omega} \omega^2 M_{21,\omega}^2 \hat{p}(\omega-k)\hat{p}(j-\omega) + i\frac{1}{N} i(k)\sum_{\omega} \omega M_{21,\omega}^2 \hat{p}(\omega-k)\hat{q}(j-\omega) + \\
&\quad i(j)i\frac{1}{N}\sum_{\omega} \omega M_{21,\omega}^2 \hat{q}(\omega-k)\hat{p}(j-\omega) + \frac{1}{N}\sum_{\omega} M_{21,\omega}^2 \hat{q}(\omega-k)\hat{q}(j-\omega) \\
&= \frac{1}{N} i(k)i(j)\sum_{\omega} \omega^2 M_{11,\omega}^2 \hat{p}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \\
&\quad \frac{1}{N} i(k)i(j)\sum_{\omega} \omega^2 M_{21,\omega}^2 \hat{p}(\omega-k)\hat{p}(j-\omega) + \\
&\quad \frac{1}{N} i(k)\sum_{\omega} i\omega M_{11,\omega}^2 \hat{p}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q}) + i\frac{1}{N} i(k)\sum_{\omega} \omega M_{21,\omega}^2 \hat{p}(\omega-k)\hat{q}(j-\omega) + \\
&\quad \frac{1}{N} i(j)\sum_{\omega} i\omega M_{11,\omega}^2 \hat{q}(\omega-k)\hat{p}(j-\omega)(\overline{p}\omega^2+\overline{q}) + i(j)i\frac{1}{N}\sum_{\omega} \omega M_{21,\omega}^2 \hat{q}(\omega-k)\hat{p}(j-\omega) + \\
&\quad \frac{1}{N} \sum_{\omega} M_{11,\omega}^2 \hat{q}(\omega-k)\hat{q}(j-\omega)(\overline{p}\omega^2+\overline{q}) + \frac{1}{N}\sum_{\omega} M_{21,\omega}^2 \hat{q}(\omega-k)\hat{q}(j-\omega) \\
&= \frac{1}{N} i(k)i(j)\sum_{\omega} \omega^2 \hat{p}(\omega-k)\hat{p}(j-\omega)\left(M_{11,\omega}^2(\overline{p}\omega^2+\overline{q}) + M_{21,\omega}^2\right) + \\
&\quad \frac{1}{N} i(k)i\sum_{\omega} \omega \hat{p}(\omega-k)\hat{q}(j-\omega)\left(M_{11,\omega}^2(\overline{p}\omega^2+\overline{q}) + M_{21,\omega}^2\right) + \\
&\quad \frac{1}{N} i(j)i\sum_{\omega} \omega \hat{q}(\omega-k)\hat{p}(j-\omega)\left(M_{11,\omega}^2(\overline{p}\omega^2+\overline{q}) + M_{21,\omega}^2\right) + \\
&\quad \frac{1}{N} \sum_{\omega} \hat{q}(\omega-k)\hat{q}(j-\omega)\left(M_{11,\omega}^2(\overline{p}\omega^2+\overline{q}) + M_{21,\omega}^2\right).
\end{aligned}
$$

To get an upper bound for $\|G_{11}\|_\infty$, we use the following bounds on P_{ij} and $M_{ij,\omega}$, which are the coefficients in the linear approximations of the various components of the solution operator,

$$
\begin{aligned}
\left|P_{11}(l_{2,\omega})\right| &\leq \left|\cos\left(\sqrt{l_{2,\omega}}\Delta t\right)\right| \leq 1, \\
\left|P_{21}(l_{2,\omega})\right| &\leq \left|-l_{2,\omega}^{1/2}\sin\left(l_{2,\omega}^{1/2}\Delta t\right)\right| \leq \left|l_{2,\omega}^{1/2}\right| l_{2,\omega}^{1/2}\Delta t = l_{2,\omega}\Delta t, \\
\left|M_{11,\omega}\right| &\leq \frac{\Delta t^2}{2}, \\
\left|M_{11,\omega}\right| &\leq \frac{\Delta t^2}{(\overline{p}\omega^2+\overline{q})^{1/2}\Delta t + (\overline{q})^{1/2}\Delta t}, \\
\left|M_{21,\omega}\right| &\leq \Delta t\left(1+\frac{2\overline{q}}{\overline{p}\omega^2}\right).
\end{aligned}
$$

We have multiple bounds for M_{11} and M_{22} so that different terms will have the same order of magnitude in terms of N and Δt. Then

$$
\begin{aligned}
\left|B_{jk}\right| &\leq \Bigg|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}k^2+\overline{q})\frac{\Delta t^2}{2} + \frac{1}{\sqrt{N}}\frac{\Delta t^2}{2}\hat{q}(j-k)(\overline{p}k^2+\overline{q}) + \\
&\quad \frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(1+\frac{2\overline{q}}{\overline{p}k^2}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(1+\frac{2\overline{q}}{\overline{p}k^2}\right)\Bigg| \\
&= \left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}k^2}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}k^2}\right)\right| \\
&\leq \left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}k^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
|C_{jk}| &\leq \Bigg|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}j^2+\overline{q})\frac{\Delta t^2}{2} + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\frac{\Delta t^2}{2} + \\
&\quad \frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(1+\frac{2\overline{q}}{\overline{p}j^2}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(1+\frac{2\overline{q}}{\overline{p}j^2}\right)\Bigg| \\
&= \left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}j^2}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}j^2}\right)\right| \\
&\leq \left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right) + \frac{1}{\sqrt{N}}\hat{q}(j-k)(\overline{p}j^2+\overline{q})\Delta t^2\left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
|D_{jk}| \leq\ & \left|\frac{1}{N} i(k) i(j) \sum_{\omega} \omega^2 \hat{p}(\omega-k)\hat{p}(j-\omega)\left(\frac{\Delta t^2}{\left((\overline{p}\omega^2+\overline{q})^{1/2}+(\overline{q})^{1/2}\right)^2}(\overline{p}\omega^2+\overline{q})+\right.\right. \\
& \left.\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}\omega^2}+\frac{4\overline{q}^2}{\overline{p}^2\omega^4}\right)\right)+\frac{1}{N} i(k) i \sum_{\omega} \omega \hat{p}(\omega-k)\hat{q}(j-\omega)\times \\
& \left(\frac{\Delta t^2}{\left((\overline{p}\omega^2+\overline{q})^{1/2}+(\overline{q})^{1/2}\right)^2}(\overline{p}\omega^2+\overline{q})+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}\omega^2}+\frac{4\overline{q}^2}{\overline{p}^2\omega^4}\right)\right)+ \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{q}(\omega-k)\hat{p}(j-\omega)\left(\frac{\Delta t^2}{\left((\overline{p}\omega^2+\overline{q})^{1/2}+(\overline{q})^{1/2}\right)^2}(\overline{p}\omega^2+\overline{q})+\right. \\
& \left.\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}\omega^2}+\frac{4\overline{q}^2}{\overline{p}^2\omega^4}\right)\right)+\frac{1}{N}\sum_{\omega} \hat{q}(\omega-k)\hat{q}(j-\omega)\left(\frac{\Delta t^2}{\left((\overline{p}\omega^2+\overline{q})^{1/2}+(\overline{q})^{1/2}\right)^2}\times\right. \\
& \left.\left.(\overline{p}\omega^2+\overline{q})+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}\omega^2}+\frac{4\overline{q}^2}{\overline{p}^2\omega^4}\right)\right)\right| \\
\leq\ & \left|\frac{1}{N} i(k) i(j) \sum_{\omega} \omega^2 \hat{p}(\omega-k)\hat{p}(j-\omega)\left(\Delta t^2+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}}+\frac{4\overline{q}^2}{\overline{p}^2}\right)\right)+\right. \\
& \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{p}(\omega-k)\hat{q}(j-\omega)\left(\Delta t^2+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}}+\frac{4\overline{q}^2}{\overline{p}^2}\right)\right)+ \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{q}(\omega-k)\hat{p}(j-\omega)\left(\Delta t^2+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}}+\frac{4\overline{q}^2}{\overline{p}^2}\right)\right)+ \\
& \left.\frac{1}{N}\sum_{\omega} \hat{q}(\omega-k)\hat{q}(j-\omega)\left(\Delta t^2+\Delta t^2\left(1+\frac{4\overline{q}}{\overline{p}}+\frac{4\overline{q}^2}{\overline{p}^2}\right)\right)\right| \\
=\ & \left|\frac{1}{N} i(k) i(j) \sum_{\omega} \omega^2 \hat{p}(\omega-k)\hat{p}(j-\omega)\left(\Delta t^2+\Delta t^2\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)+\right. \\
& \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{p}(\omega-k)\hat{q}(j-\omega)\left(\Delta t^2+\Delta t^2\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)+ \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{q}(\omega-k)\hat{p}(j-\omega)\left(\Delta t^2+\Delta t^2\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)+ \\
& \left.\frac{1}{N}\sum_{\omega} \hat{q}(\omega-k)\hat{q}(j-\omega)\left(\Delta t^2+\Delta t^2\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)\right|
\end{aligned}
$$

From these bounds, we obtain

$$
\begin{aligned}
\|G_{11}\|_\infty \leq\ & \max_{1\leq j\leq N} \sum_{k=1}^{N} \left|A_{jk}+B_{jk}+C_{jk}+D_{jk}\right| \\
\leq\ & \max_{1\leq j\leq N} (\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}j^2+\bar{q})^{-1/2}(\overline{p}j^2+\overline{q})+ \\
& \sum_{k=1}^{N} \left|\frac{1}{\sqrt{N}} jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}(\overline{p}k^2+\overline{q})\Delta t^2 \left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right)\right| + \\
& \sum_{k=1}^{N} \left|\frac{1}{\sqrt{N}} \hat{q}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}(\overline{p}k^2+\overline{q})\Delta t^2 \left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right)\right| + \\
& \sum_{k=1}^{N} \left|\frac{1}{\sqrt{N}} jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}(\overline{p}j^2+\overline{q})\Delta t^2 \left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right)\right| + \\
& \sum_{k=1}^{N} \left|\frac{1}{\sqrt{N}} \hat{q}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}(\overline{p}j^2+\overline{q})\Delta t^2 \left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right)\right| + \\
& \sum_{k=1}^{N} \left|\frac{1}{N} i(k)i(j)(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}\sum_{\omega}\omega^2\hat{p}(\omega-k)\hat{p}(j-\omega)\Delta t^2 \times \right. \\
& \left. \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)\right| + \sum_{k=1}^{N} \left|\frac{1}{N} i(k)i(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}\sum_{\omega}\omega\hat{p}(\omega-k)\times \right. \\
& \left. \hat{q}(j-\omega)\Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)\right| + \sum_{k=1}^{N} \left|\frac{1}{N} i(j)i(\bar{p}j^2+\bar{q})^{-1/2}(\bar{p}k^2+\bar{q})^{-1/2}\times \right. \\
& \left. \sum_{\omega}\omega\hat{q}(\omega-k)\hat{p}(j-\omega)\Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)\right| + \sum_{k=1}^{N} \left|\frac{1}{N}(\bar{p}j^2+\bar{q})^{-1/2}\times \right. \\
& \left. (\bar{p}k^2+\bar{q})^{-1/2}\sum_{\omega}\hat{q}(\omega-k)\hat{q}(j-\omega)\Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \max_{1\leq j\leq N} 1 + j\Delta t^2 \left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right) \|\tilde{p}\|_\infty (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|k(\overline{p}k^2+\overline{q})^{1/2}\right| + \\
& \Delta t^2 \left(\frac{3}{2}+\frac{\overline{2q}}{\overline{p}}\right) \|\tilde{q}\|_\infty (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|(\overline{p}k^2+\overline{q})^{1/2}\right| + \\
& j\Delta t^2 \|\tilde{p}\|_\infty \left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right) (\overline{p}j^2+\overline{q})^{1/2} \sum_{k=1}^{N} \left|k(\overline{p}k^2+\overline{q})^{-1/2}\right| + \\
& \Delta t^2 \left(\frac{3}{2}+\frac{2\overline{q}}{\overline{p}}\right) \|\tilde{q}\|_\infty (\overline{p}j^2+\overline{q})^{1/2} \sum_{k=1}^{N} \left|(\overline{p}k^2+\overline{q})^{-1/2}\right| + \\
& j\Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right) \|\tilde{p}\|_\infty^2 (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|k(\overline{p}k^2+\overline{q})^{-1/2}\sum_{\omega}\omega^2\right| + \\
& \Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right) \|\tilde{p}\|_\infty \|\tilde{q}\|_\infty (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|k(\overline{p}k^2+\overline{q})^{-1/2}\sum_{\omega}\omega\right| + \\
& j\Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right) \|\tilde{p}\|_\infty \|\tilde{q}\|_\infty (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|(\overline{p}k^2+\overline{q})^{-1/2}\sum_{\omega}\omega\right| + \\
& \Delta t^2 \left(1+\left(\frac{2\overline{q}}{\overline{p}}+1\right)^2\right) \|\tilde{q}\|_\infty^2 (\overline{p}j^2+\overline{q})^{-1/2} \sum_{k=1}^{N} \left|(\overline{p}k^2+\overline{q})^{-1/2}\sum_{\omega}\right| . \qquad (3.9)
\end{aligned}
$$

We can bound each of the summations in (3.9) as in the examples below. Because p and q are bandlimited,

$$\hat{p}(j-k)\sum_{k=1}^{N}|k| = \hat{p}(j-k)\sum_{k=j-\omega_{\max},k\neq j}^{j+\omega_{\max}}|k| \leq N^{1/2}\|\tilde{p}\|_\infty \sum_{k=j-\omega_{\max},k\neq j}^{j+\omega_{\max}}|k|. \qquad (3.10)$$

Let $\eta = j-k$, so that $k = j-\eta$. If $0\leq j\leq \omega_{\max}$, we have

$$\sum_{\eta=-\omega_{\max}}^{\omega_{\max}} |j-\eta| = \sum_{\eta=1}^{\omega_{\max}} |j-\eta|+|j+\eta| = \sum_{\eta=1}^{\omega_{max}} 2\eta = \omega_{\max}^2+\omega_{\max} \leq 2\omega_{\max}^2. \qquad (3.11)$$

If $j\geq \omega_{\max}$, then

$$\sum_{\eta=-\omega_{\max}}^{\omega_{\max}} |j-\eta| = \sum_{\eta=1}^{\omega_{\max}} |j-\eta|+|j+\eta| = \sum_{\eta=1}^{\omega_{\max}} 2j = 2j\omega_{\max}. \qquad (3.12)$$

Taking the maximum of (3.11) and (3.12), we get

$$\max\{2\omega_{\max}^2, 2j\omega_{\max}\} = 2\omega_{\max}\max\{\omega_{\max}, j\} = 2j\omega.$$

Therefore, the summation in (3.10) is $O(N)$.

As another example, if we have $\hat{q}(j-k)$ where $|j-k| \leq \omega_{\max}$, then

$$\hat{q}(j-k)(\overline{p}j^2+\overline{q})^{1/2}\sum_{k=1}^{N}\left|(\overline{p}k^2+\overline{q})^{-1/2}\right| = \hat{q}(j-k)(\overline{p}j^2+\overline{q})^{1/2}\sum_{k=j-\omega_{\max},k\neq j}^{j+\omega_{\max}}\left|(\overline{p}k^2+\overline{q})^{-1/2}\right|.$$

Assume $j \geq 0$. For $j \geq \omega_{\max}$, we have

$$\begin{aligned}(\overline{p}j^2+\overline{q})^{1/2}\sum_{k=j-\omega_{\max}k\neq j}^{j+\omega_{\max}}\left|(\overline{p}k^2+\overline{q})^{-1/2}\right| &\leq (\overline{p}j^2+\overline{q})^{1/2}\sum_{k=j-\omega_{max}k\neq j}^{j+\omega_{max}}\frac{1}{(\overline{p}(j-\omega_{max})^2+\overline{q})^{1/2}}\\ &\leq (\overline{p}j^2+\overline{q})^{1/2}\frac{2\omega_{max}}{(\overline{p}(j-\omega_{max})^2+\overline{q})^{1/2}}.\end{aligned}$$

As $j \to \infty$, from

$$\left(\frac{2\omega_{max}(\overline{p}j^2+\overline{q})^{1/2}}{(\overline{p}(j-\omega_{max})^2+\overline{q})^{1/2}}\right)^2 = \frac{4\omega_{max}^2(\overline{p}j^2+\overline{q})}{(\overline{p}(j-\omega_{max})^2+\overline{q})}$$

we obtain

$$\lim_{j\to\infty}\frac{4\omega_{max}^2(\overline{p}j^2+\overline{q})}{\overline{p}(j-\omega_{max})^2+\overline{q}} = \lim_{j\to\infty}\frac{4\omega_{max}^2(\overline{p}j)}{2\overline{p}(j-\omega_{max})} = \lim_{j\to\infty}\frac{8\omega_{max}^2(\overline{p})}{2\overline{p}} = 4\omega_{max}^2.$$

Therefore,

$$\lim_{j\to\infty}\left(\frac{2\omega_{max}(\overline{p}j^2+\overline{q})^{1/2}}{(\overline{p}(j-\omega_{max})^2+\overline{q})^{1/2}}\right)^2 = 2\omega_{max}.$$

For $0 \leq j \leq \omega_{\max}$, we have

$$\begin{aligned}(\overline{p}j^2+\overline{q})^{1/2}\sum_{k=1}^{N}\left|(\overline{p}k^2+\overline{q})^{-1/2}\right| &= (\overline{p}j^2+\overline{q})^{1/2}\sum_{k=j-\omega_{max}}^{j+\omega_{max}}\left|(\overline{p}k^2+\overline{q})^{-1/2}\right|\\ &\leq (\overline{p}j^2+\overline{q})^{1/2}\sum_{k=j-\omega_{max}}^{j+\omega_{max}}\frac{1}{(\overline{p}(0)^2+\overline{q})^{1/2}}\\ &\leq (\overline{p}j^2+\overline{q})^{1/2}\frac{2\omega_{max}}{\overline{q}^{1/2}}\\ &\leq \frac{2\omega_{max}(\overline{p}\omega_{max}^2+\overline{q})^{1/2}}{(\overline{q})^{1/2}}.\end{aligned}$$

Therefore, the sum can be bounded independently of N.

Using a similar approach to evaluate the remaining summations in (3.9), we obtain

$$\begin{aligned}\|G_{11}\|_\infty &\leq 1+C_{11,p}\|\tilde{p}\|_\infty\Delta t^2N^2+C_{11,q}\|\tilde{q}\|_\infty\Delta t^2+C_{11,p}\|\tilde{p}\|_\infty\Delta t^2N^2+C_{11,q}\|\tilde{q}\|_\infty\Delta t^2+\\ &\quad C_{11,p^2}\|\tilde{p}\|_\infty^2\Delta t^2N^2+C_{11,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty\Delta t^2+C_{11,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty\Delta t^2+C_{11,q^2}\|\tilde{q}\|_\infty^2\Delta t^2\\ &\leq 1+C_{11,p}\|\tilde{p}\|_\infty\Delta t^2N^2+C_{11,q}\|\tilde{q}\|_\infty\Delta t^2+C_{11,p^2}\|\tilde{p}\|_\infty^2\Delta t^2N^2+\\ &\quad C_{11,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty\Delta t^2+C_{11,q^2}\|\tilde{q}\|_\infty^2\Delta t^2.\end{aligned}$$

□

Using the same approach, we find the matrix G_{22} defined in (3.4) satisfies a similar bound as G_{11}, with unspecified constant factors.

Lemma 3.2.2. *Assume $\hat{p}(\omega) = 0$ and $\hat{q}(\omega) = 0$ for $|\omega| \geq \omega_{\max}$. Then the matrix G_{12} defined in (3.4) satisfies*

$$\begin{aligned} \|G_{12}\|_\infty \quad \leq \quad & C_{12,p}\|\tilde{p}\|_\infty \Delta t N + C_{12,q}\|\tilde{q}\|_\infty \Delta t + C_{12,p^2}\|\tilde{p}\|_\infty^2 \Delta t N + \\ & C_{12,pq}\|\tilde{p}\|_\infty \|\tilde{q}\|_\infty \Delta t + C_{12,q^2}\|\tilde{q}\|_\infty^2 \Delta t \end{aligned} \tag{3.13}$$

where each constant is independent of N and Δt.

Proof. From (3.5) we have

$$[\mathbf{u}^n]^T G_{12} \mathbf{u}_t^n = \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)} \hat{\mathbf{z}}_{12}(\omega)(\bar{p}\omega^2 + \bar{q}) + \sum_\omega \overline{\hat{\mathbf{z}}_{21}(\omega)} \hat{\mathbf{z}}_{22}(\omega).$$

Again, we simplify the first and second terms separately, then combine them. We have

$$\begin{aligned} \overline{\hat{\mathbf{z}}_{11}(\omega)} \hat{\mathbf{z}}_{12}(\omega) \quad = \quad & \Bigg(P_{11}(l_{2,\omega})\hat{u}(-\omega) - i\omega M_{11,\omega} \frac{1}{\sqrt{N}} \sum_j [\hat{p}(j-\omega) i(j) \hat{u}(-j)] + \\ & M_{11,\omega} \frac{1}{\sqrt{N}} \sum_j \hat{q}(j-\omega)\hat{u}(-j) \Bigg) \Bigg(P_{12}(l_{2,\omega})\hat{u}_t(\omega) - i\omega M_{12,\omega} \frac{1}{\sqrt{N}} \times \\ & \sum_k [\hat{p}(\omega-k) i(k) \hat{u}_t(k)] + M_{12,\omega} \frac{1}{\sqrt{N}} \sum_k \hat{q}(\omega-k)\hat{u}_t(k) \Bigg) \\ = \quad & P_{11}(l_{2,\omega})\hat{u}(-\omega) P_{12}(l_{2,\omega})\hat{u}_t(\omega) - P_{11}(l_{2,\omega})\hat{u}(-\omega) i\omega M_{12,\omega} \frac{1}{\sqrt{N}} \times \\ & \sum_k [\hat{p}(\omega-k) i(k) \hat{u}_t(k)] + P_{11}(l_{2,\omega})\hat{u}(-\omega) M_{12,\omega} \frac{1}{\sqrt{N}} \sum_k \hat{q}(\omega-k)\hat{u}_t(k) - \\ & i\omega M_{11,\omega} \frac{1}{\sqrt{N}} \sum_j [\hat{p}(j-\omega) i(j) \hat{u}(-j)] P_{12}(l_{2,\omega})\hat{u}_t(\omega) - \\ & \omega^2 M_{11,\omega} M_{12,\omega} \frac{1}{N} \sum_j [\hat{p}(j-\omega) i(j) \hat{u}(-j)] \sum_k [\hat{p}(\omega-k) i(k) \hat{u}_t(k)] - \\ & i\omega M_{11,\omega} M_{12,\omega} \frac{1}{N} \sum_j [\hat{p}(j-\omega) i(j) \hat{u}(-j)] \sum_k \hat{q}(\omega-k)\hat{u}_t(k) + \\ & M_{11,\omega} \frac{1}{\sqrt{N}} \sum_j \hat{q}(j-\omega)\hat{u}(-j) P_{12}(l_{2,\omega})\hat{u}_t(\omega) - \\ & i\omega M_{11,\omega} M_{12,\omega} \frac{1}{N} \sum_j \hat{q}(j-\omega)\hat{u}(-j) \sum_k [\hat{p}(\omega-k) i(k) \hat{u}_t(k)] + \\ & M_{11,\omega} M_{12,\omega} \frac{1}{N} \sum_j \hat{q}(j-\omega)\hat{u}(-j) \sum_k \hat{q}(\omega-k)\hat{u}_t(k). \end{aligned}$$

Let $s_2 = \sum_\omega \overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{12}(\omega)(\bar{p}\omega^2+\bar{q})$. We have

$$\begin{aligned}
s_2 \;=\;& \sum_\omega P_{11}(l_{2,\omega})\hat{u}(-\omega)P_{12}(l_{2,\omega})\hat{u}_t(\omega)(\bar{p}\omega^2+\bar{q}) - \\
& \sum_\omega\sum_k \hat{u}(-\omega)\hat{u}_t(k)\left[P_{11}(l_{2,\omega})i\omega M_{12,\omega}\frac{1}{\sqrt{N}}\hat{p}(\omega-k)i(k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_\omega\sum_k \hat{u}(-\omega)\hat{u}_t(k)\left[P_{11}(l_{2,\omega})M_{12,\omega}\frac{1}{\sqrt{N}}\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j\sum_\omega \hat{u}(-j)\hat{u}_t(\omega)\left[i\omega M_{11,\omega}\frac{1}{\sqrt{N}}\hat{p}(j-\omega)i(j)P_{12}(l_{2,\omega})(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega \omega^2 M_{11,\omega}M_{12,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)\hat{p}(\omega-k)i(k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega i\omega M_{11,\omega}M_{12,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_j\sum_\omega \hat{u}(-j)\hat{u}_t(\omega)\left[M_{11,\omega}\frac{1}{\sqrt{N}}\hat{q}(j-\omega)P_{12}(l_{2,\omega})(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega i\omega M_{11,\omega}M_{12,\omega}\frac{1}{N}\hat{q}(j-\omega)\hat{p}(\omega-k)i(k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega M_{11,\omega}M_{12,\omega}\frac{1}{N}\hat{q}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] \\
=\;& \sum_j \hat{u}(-j)\hat{u}_t(j)\left[P_{11}(l_{2,j})P_{12}(l_{2,j})(\bar{p}j^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}ijP_{11}(l_{2,j})M_{12,j}\hat{p}(j-k)i(k)(\bar{p}j^2+\bar{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{12,j}\hat{q}(j-k)(\bar{p}j^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}i(j)ikM_{11,k}\hat{p}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(j)i(k)\sum_\omega \omega^2 M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(j)i\sum_\omega \omega M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}M_{11,k}\hat{q}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right] - \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(k)i\sum_\omega \omega M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_j\sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}\sum_\omega M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right]. \qquad (3.14)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{22}(\omega) &= \Bigg(P_{21}(l_{2,\omega})\hat{u}(-\omega) - i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)] + \\
&\quad M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j \hat{q}(j-\omega)\hat{u}(-j)\Bigg)\Bigg(P_{22}(l_{2,\omega})\hat{u}_t(\omega) - i\omega M_{22,\omega}\frac{1}{\sqrt{N}}\times \\
&\quad \sum_k[\hat{p}(\omega-k)i(k)\hat{u}_t(k)] + M_{22,\omega}\frac{1}{\sqrt{N}}\sum_k \hat{q}(\omega-k)\hat{u}_t(k)\Bigg) \\
&= P_{21}(l_{2,\omega})\hat{u}(-\omega)P_{22}(l_{2,\omega})\hat{u}_t(\omega) - \\
&\quad P_{21}(l_{2,\omega})\hat{u}(-\omega)i\omega M_{22,\omega}\frac{1}{\sqrt{N}}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}_t(k)] + \\
&\quad P_{21}(l_{2,\omega})\hat{u}(-\omega)M_{22,\omega}\frac{1}{\sqrt{N}}\sum_k \hat{q}(\omega-k)\hat{u}_t(k) - \\
&\quad i\omega M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)]P_{22}(l_{2,\omega})\hat{u}_t(\omega) - \\
&\quad \omega^2 M_{21,\omega}\frac{1}{N}\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)]M_{22,\omega}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}_t(k)] - \\
&\quad i\omega M_{21,\omega}\frac{1}{N}\sum_j[\hat{p}(j-\omega)i(j)\hat{u}(-j)]M_{22,\omega}\sum_k \hat{q}(\omega-k)\hat{u}_t(k) + \\
&\quad M_{21,\omega}\frac{1}{\sqrt{N}}\sum_j \hat{q}(j-\omega)\hat{u}(-j)P_{22}(l_{2,\omega})\hat{u}_t(\omega) - \\
&\quad M_{21,\omega}\frac{1}{N}\sum_j \hat{q}(j-\omega)\hat{u}(-j)i\omega M_{22,\omega}\sum_k[\hat{p}(\omega-k)i(k)\hat{u}_t(k)] + \\
&\quad M_{21,\omega}\frac{1}{N}\sum_j \hat{q}(j-\omega)\hat{u}(-j)M_{22,\omega}\sum_k \hat{q}(\omega-k)\hat{u}_t(k)
\end{aligned}$$

which yields

$$
\begin{aligned}
\sum_{\omega}\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{22}(\omega) &= \sum_{\omega} P_{21}(l_{2,\omega})\hat{u}(-\omega)P_{22}(l_{2,\omega})\hat{u}_t(\omega) - \\
&\sum_{\omega}\sum_{k}\hat{u}(-\omega)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,\omega})i\omega M_{22,\omega}\hat{p}(\omega-k)i(k)\right] + \\
&\sum_{\omega}\sum_{k}\hat{u}(-\omega)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,\omega})M_{22,\omega}\hat{q}(\omega-k)\right] - \\
&\sum_{j}\sum_{\omega}\hat{u}(-j)\hat{u}_t(\omega)\left[\frac{1}{\sqrt{N}}i\omega M_{21,\omega}\hat{p}(j-\omega)i(j)P_{22}(l_{2,\omega})\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}\omega^2 M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{p}(\omega-k)i(k)\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}i\omega M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{q}(\omega-k)\right] + \\
&\sum_{j}\sum_{\omega}\hat{u}(-j)\hat{u}_t(\omega)\left[M_{21,\omega}\frac{1}{\sqrt{N}}\hat{q}(j-\omega)P_{22}(l_{2,\omega})\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)i\omega M_{22,\omega}\hat{p}(\omega-k)i(k)\right] + \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)M_{22,\omega}\hat{q}(\omega-k)\right] \\
&= \sum_{j}\hat{u}(-j)\hat{u}_t(j)\left[P_{21}(l_{2,j})P_{22}(l_{2,j})\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})ijM_{22,j}\hat{p}(j-k)i(k)\right] + \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{22,j}\hat{q}(j-k)\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}ikM_{21,k}\hat{p}(j-k)i(j)P_{22}(l_{2,k})\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}\omega^2 M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{p}(\omega-k)i(k)\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}i\omega M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{q}(\omega-k)\right] + \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[M_{21,k}\frac{1}{\sqrt{N}}\hat{q}(j-k)P_{22}(l_{2,k})\right] - \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)i\omega M_{22,\omega}\hat{p}(\omega-k)i(k)\right] + \\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)M_{22,\omega}\hat{q}(\omega-k)\right]. \qquad (3.15)
\end{aligned}
$$

Putting the two equations (3.14) and (3.15) together, we obtain

$$\begin{aligned}
&\sum_{\omega}\overline{\hat{\mathbf{z}}_{11}(\omega)}\hat{\mathbf{z}}_{12}(\omega)(\bar{p}\omega^2+\bar{q})+\sum_{\omega}\overline{\hat{\mathbf{z}}_{21}(\omega)}\hat{\mathbf{z}}_{22}(\omega)\\
= \; &\sum_{j}P_{11}(l_{2,j})\hat{u}(-j)P_{12}(l_{2,j})\hat{u}_t(j)(\bar{p}j^2+\bar{q})-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}ijP_{11}(l_{2,j})M_{12,j}\hat{p}(j-k)i(k)(\bar{p}j^2+\bar{q})\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{12,j}\hat{q}(j-k)(\bar{p}j^2+\bar{q})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}i(j)ikM_{11,k}\hat{p}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(j)i(k)\sum_{\omega}\omega^2M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(j)i\sum_{\omega}\omega M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}M_{11,k}\hat{q}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}i(k)i\sum_{\omega}\omega M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N}\sum_{\omega}M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right]\\
&\sum_{j}P_{21}(l_{2,j})\hat{u}(-j)P_{22}(l_{2,j})\hat{u}_t(j)-\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})ijM_{22,j}\hat{p}(j-k)i(k)\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{22,j}\hat{q}(j-k)\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}}ikM_{21,k}\hat{p}(j-k)i(j)P_{22}(l_{2,k})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}\omega^2M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{p}(\omega-k)i(k)\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}i\omega M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{q}(\omega-k)\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[M_{21,k}\frac{1}{\sqrt{N}}\hat{q}(j-k)P_{22}(l_{2,k})\right]-\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)i\omega M_{22,\omega}\hat{p}(\omega-k)i(k)\right]+\\
&\sum_{j}\sum_{k}\hat{u}(-j)\hat{u}_t(k)\left[\sum_{\omega}M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)M_{22,\omega}\hat{q}(\omega-k)\right]
\end{aligned}$$

$$
\begin{aligned}
= \quad & \sum_j P_{11}(l_{2,j})\hat{u}(-j)P_{12}(l_{2,j})\hat{u}_t(j)(\bar{p}j^2+\bar{q}) + \sum_j P_{21}(l_{2,j})\hat{u}(-e)P_{22}(l_{2,j})\hat{u}_t(j) - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} i(j)ikM_{11,k}\hat{p}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} M_{11,k}\hat{q}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} iki(j)M_{21,k}\hat{p}(j-k)P_{22}(l_{2,k})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[M_{21,k}\frac{1}{\sqrt{N}}\hat{q}(j-k)P_{22}(l_{2,k})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} ijP_{11}(l_{2,j})M_{12,j}\hat{p}(j-k)i(k)(\bar{p}j^2+\bar{q})\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} P_{11}(l_{2,j})M_{12,j}\hat{q}(j-k)(\bar{p}j^2+\bar{q})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} P_{21}(l_{2,j})ijM_{22,j}\hat{p}(j-k)i(k)\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{\sqrt{N}} P_{21}(l_{2,j})M_{22,j}\hat{q}(j-k)\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N} i(j)i(k)\sum_\omega \omega^2 M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N} i(j)i\sum_\omega \omega M_{11,\omega}M_{12,\omega}\hat{p}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N} i(k)i\sum_\omega \omega M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\frac{1}{N} \sum_\omega M_{11,\omega}M_{12,\omega}\hat{q}(j-\omega)\hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q})\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega \omega^2 M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{p}(\omega-k)i(k)\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega i\omega M_{21,\omega}\frac{1}{N}\hat{p}(j-\omega)i(j)M_{22,\omega}\hat{q}(\omega-k)\right] - \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)i\omega M_{22,\omega}\hat{p}(\omega-k)i(k)\right] + \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}_t(k)\left[\sum_\omega M_{21,\omega}\frac{1}{N}\hat{q}(j-\omega)M_{22,\omega}\hat{q}(\omega-k)\right] \\
& \sum_j \sum_k \hat{u}(-j)\hat{u}(k)\left[A+B+C+D\right]_{jk}
\end{aligned}
$$

where

$$
\begin{aligned}
A_{jj} &= P_{11}(l_{2,j})P_{12}(l_{2,j})(\bar{p}j^2+\bar{q})+P_{21}(l_{2,j})P_{22}(l_{2,j}) \\
&= \cos\left((\overline{p}j^2+\overline{q})^{1/2}\Delta t\right)(\overline{p}j^2+q)^{-1/2}\sin\left(\sqrt{\overline{p}j^2+q}\Delta t\right)(\bar{p}j^2+\bar{q})+ \\
&\quad \left(-(\overline{p}j^2+\overline{q})^{1/2}\sin\left((\overline{p}j^2+\overline{q})^{1/2}\Delta t\right)\cos\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right)\right) \\
&= \cos\left((\overline{p}j^2+\overline{q})^{1/2}\Delta t\right)(\overline{p}j^2+q)^{1/2}\sin\left(\sqrt{\overline{p}j^2+q}\Delta t\right)- \\
&\quad (\overline{p}j^2+\overline{q})^{1/2}\sin\left((\overline{p}j^2+\overline{q})^{1/2}\Delta t\right)\cos\left(\sqrt{\overline{p}j^2+\overline{q}}\Delta t\right) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
B_{jk} &= \frac{1}{\sqrt{N}}i(j)ikM_{11,k}\hat{p}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})+\frac{1}{\sqrt{N}}M_{11,k}\hat{q}(j-k)P_{12}(l_{2,k})(\bar{p}k^2+\bar{q})+ \\
&\quad \frac{1}{\sqrt{N}}i(j)ikM_{21,k}\hat{p}(j-k)P_{22}(l_{2,k})+M_{21,k}\frac{1}{\sqrt{N}}\hat{q}(j-k)P_{22}(l_{2,k}),
\end{aligned}
$$

$$
\begin{aligned}
C_{jk} &= \frac{1}{\sqrt{N}}ijP_{11}(l_{2,j})M_{12,j}\hat{p}(j-k)i(k)(\bar{p}j^2+\bar{q})+\frac{1}{\sqrt{N}}P_{11}(l_{2,j})M_{12,j}\hat{q}(j-k)(\bar{p}j^2+\bar{q})+ \\
&\quad \frac{1}{\sqrt{N}}P_{21}(l_{2,j})ijM_{22,j}\hat{p}(j-k)i(k)+\frac{1}{\sqrt{N}}P_{21}(l_{2,j})M_{22,j}\hat{q}(j-k),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{jk} &= \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 M_{11,\omega} M_{12,\omega} \hat{p}(j-\omega) \hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} i(j) i \sum_{\omega} \omega M_{11,\omega} M_{12,\omega} \hat{p}(j-\omega) \hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} i(k) i \sum_{\omega} \omega M_{11,\omega} M_{12,\omega} \hat{q}(j-\omega) \hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} \sum_{\omega} M_{11,\omega} M_{12,\omega} \hat{q}(j-\omega) \hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \sum_{\omega} \omega^2 M_{21,\omega} \frac{1}{N} \hat{p}(j-\omega) i(j) M_{22,\omega} \hat{p}(\omega-k) i(k) + \\
&\quad \sum_{\omega} i\omega M_{21,\omega} \frac{1}{N} \hat{p}(j-\omega) i(j) M_{22,\omega} \hat{q}(\omega-k) + \\
&\quad \sum_{\omega} M_{21,\omega} \frac{1}{N} \hat{q}(j-\omega) i\omega M_{22,\omega} \hat{p}(\omega-k) i(k) + \\
&\quad \sum_{\omega} M_{21,\omega} \frac{1}{N} \hat{q}(j-\omega) M_{22,\omega} \hat{q}(\omega-k) \\
&= \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 M_{11,\omega} M_{12,\omega} \hat{p}(j-\omega) \hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 M_{21,\omega} \hat{p}(j-\omega) M_{22,\omega} \hat{p}(\omega-k) + \\
&\quad \frac{1}{N} i(j) i \sum_{\omega} \omega M_{11,\omega} M_{12,\omega} \hat{p}(j-\omega) \hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} i(j) i \sum_{\omega} \omega M_{21,\omega} \hat{p}(j-\omega) M_{22,\omega} \hat{q}(\omega-k) + \\
&\quad \frac{1}{N} i(k) i \sum_{\omega} \omega M_{11,\omega} M_{12,\omega} \hat{q}(j-\omega) \hat{p}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \\
&\quad \frac{1}{N} i(k) i \sum_{\omega} M_{21,\omega} \hat{q}(j-\omega) \omega M_{22,\omega} \hat{p}(\omega-k) + \\
&\quad \frac{1}{N} \sum_{\omega} M_{11,\omega} M_{12,\omega} \hat{q}(j-\omega) \hat{q}(\omega-k)(\bar{p}\omega^2+\bar{q}) + \frac{1}{N} \sum_{\omega} M_{21,\omega} \hat{q}(j-\omega) M_{22,\omega} \hat{q}(\omega-k) \\
&= \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 \hat{p}(j-\omega) \hat{p}(\omega-k) \left(M_{11,\omega} M_{12,\omega} (\bar{p}\omega^2+\bar{q}) + M_{21,\omega} M_{22,\omega} \right) + \\
&\quad \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{p}(j-\omega) \hat{q}(\omega-k) \left(M_{11,\omega} M_{12,\omega} (\bar{p}\omega^2+\bar{q}) + M_{21,\omega} M_{22,\omega} \right) + \\
&\quad \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{q}(j-\omega) \hat{p}(\omega-k) \left(M_{11,\omega} M_{12,\omega} (\bar{p}\omega^2+\bar{q}) + M_{21,\omega} M_{22,\omega} \right) + \\
&\quad \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega) \hat{q}(\omega-k) \left(M_{11,\omega} M_{12,\omega} (\bar{p}\omega^2+\bar{q}) + M_{21,\omega} M_{22,\omega} \right).
\end{aligned}
$$

To get an upper bound for $\|G_{12}\|_\infty$, we use the following bounds on P_{ij} and $M_{ij,\omega}$:

$$
\begin{aligned}
|P_{12}(l_{2,\omega})| &\leq \left| l_{2,\omega}^{-1/2} \sin(l_{2,\omega}^{1/2}\Delta t) \right| \leq \left| l_{2,\omega}^{-1/2} \right| l_{2,\omega}^{1/2}\Delta t = \Delta t, \\
|M_{11,\omega}| &= |M_{22,\omega}| \leq \frac{2}{\overline{p}\omega^2}, \\
|M_{12,\omega}| &\leq \frac{2\Delta t}{\overline{p}\omega^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|B_{jk}| &\leq \Bigg| \frac{1}{\sqrt{N}} jk \frac{2}{\bar{p}k^2} \hat{p}(j-k)\Delta t(\bar{p}k^2+\bar{q}) + \frac{1}{\sqrt{N}} \frac{2}{\bar{p}k^2} \hat{q}(j-k)\Delta t(\bar{p}k^2+\bar{q}) + \\
&\quad \frac{1}{\sqrt{N}} jk\Delta t \left(1+\frac{2\overline{q}}{\overline{p}k^2}\right) \hat{p}(j-k) + \Delta t \left(1+\frac{2\overline{q}}{\overline{p}k^2}\right) \frac{1}{\sqrt{N}} \hat{q}(j-k) \Bigg| \\
&= \Bigg| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)\Delta t \left(\frac{2}{\bar{p}k^2}(\bar{p}k^2+\bar{q}) + \left(1+\frac{2\overline{q}}{\overline{p}k^2}\right)\right) + \\
&\quad \frac{1}{\sqrt{N}} \hat{q}(j-k)\Delta t \left(\frac{2}{\bar{p}k^2}(\bar{p}k^2+\bar{q}) + \left(1+\frac{2\overline{q}}{\overline{p}k^2}\right)\right) \Bigg| \\
&= \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)\Delta t \left(3+\frac{4\overline{q}}{\overline{p}k^2}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)\Delta t \left(3+\frac{4\overline{q}}{\overline{p}k^2}\right) \right| \\
&\leq \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)\Delta t \left(3+\frac{4\overline{q}}{\overline{p}}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)\Delta t \left(3+\frac{4\overline{q}}{\overline{p}}\right) \right|,
\end{aligned}
$$

$$
\begin{aligned}
|C_{jk}| &\leq \Bigg| \frac{1}{\sqrt{N}} j \frac{2\Delta t}{\bar{p}j^2} \hat{p}(j-k)k(\bar{p}j^2+\bar{q}) + \frac{1}{\sqrt{N}} \frac{2\Delta t}{\bar{p}j^2} \hat{q}(j-k)(\bar{p}j^2+\bar{q}) + \\
&\quad \frac{1}{\sqrt{N}} (\bar{p}j^2+\bar{q})\Delta t\, j \frac{2}{\bar{p}j^2} \hat{p}(j-k)k + \frac{1}{\sqrt{N}} (\bar{p}j^2+\bar{q})\Delta t \frac{2}{\bar{p}j^2} 2\hat{q}(j-k) \Bigg| \\
&= \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{2}{\bar{p}j^2}+\frac{2}{\bar{p}j^2}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{2}{\bar{p}j^2}+\frac{2}{\bar{p}j^2}\right) \right| \\
&\leq \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{4}{\bar{p}j^2}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{4}{\bar{p}j^2}\right) \right| \\
&\leq \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{2}{\bar{p}j^2}+\frac{2}{\bar{p}j^2}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)(\bar{p}j^2+\bar{q})\Delta t \left(\frac{2}{\bar{p}j^2}+\frac{2}{\bar{p}j^2}\right) \right| \\
&\leq \left| \frac{1}{\sqrt{N}} jk\hat{p}(j-k)\Delta t \left(4+\frac{4\bar{q}}{\bar{p}}\right) + \frac{1}{\sqrt{N}} \hat{q}(j-k)\Delta t \left(4+\frac{4\bar{q}}{\bar{p}}\right) \right|,
\end{aligned}
$$

and

$$
\begin{aligned}
|D_{jk}| \leq & \left| \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 \hat{p}(j-\omega) \hat{p}(\omega-k) \left(\frac{2}{\bar{p}\omega^2} \frac{2\Delta t}{\bar{p}\omega^2} (\bar{p}\omega^2 + \bar{q}) + \Delta t \left(1 + \frac{2\bar{q}}{\bar{p}\omega^2} \right) \frac{2}{\bar{p}\omega^2} \right) + \right. \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{p}(j-\omega) \hat{q}(\omega-k) \left(\frac{2}{\bar{p}\omega^2} \frac{2\Delta t}{\bar{p}\omega^2} (\bar{p}\omega^2 + \bar{q}) + \Delta t \left(1 + \frac{2\bar{q}}{\bar{p}\omega^2} \right) \frac{2}{\bar{p}\omega^2} \right) + \\
& \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{q}(j-\omega) \hat{p}(\omega-k) \left(\frac{2}{\bar{p}\omega^2} \frac{2\Delta t}{\bar{p}\omega^2} (\bar{p}\omega^2 + \bar{q}) + \Delta t \left(1 + \frac{2\bar{q}}{\bar{p}\omega^2} \right) \frac{2}{\bar{p}\omega^2} \right) + \\
& \left. \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega) \hat{q}(\omega-k) \left(\frac{2}{\bar{p}\omega^2} \frac{2\Delta t}{\bar{p}\omega^2} (\bar{p}\omega^2 + \bar{q}) + \Delta t \left(1 + \frac{2\bar{q}}{\bar{p}\omega^2} \right) \frac{2}{\bar{p}\omega^2} \right) \right| \\
= & \left| \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 \hat{p}(j-\omega) \hat{p}(\omega-k) \left(\frac{4\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} + \frac{2\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} \right) + \right. \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{p}(j-\omega) \hat{q}(\omega-k) \left(\frac{4\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} + \frac{2\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} \right) + \\
& \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{q}(j-\omega) \hat{p}(\omega-k) \left(\frac{4\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} + \frac{2\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} \right) + \\
& \left. \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega) \hat{q}(\omega-k) \left(\frac{4\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} + \frac{2\Delta t}{\bar{p}\omega^2} + \frac{4\Delta t \bar{q}}{(\bar{p}\omega^2)^2} \right) \right| \\
= & \left| \frac{1}{N} i(j) i(k) \sum_{\omega} \omega^2 \hat{p}(j-\omega) \hat{p}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{(\bar{p}\omega^2)^2} \right) + \right. \\
& \frac{1}{N} i(j) i \sum_{\omega} \omega \hat{p}(j-\omega) \hat{q}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{(\bar{p}\omega^2)^2} \right) + \\
& \frac{1}{N} i(k) i \sum_{\omega} \omega \hat{q}(j-\omega) \hat{p}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{(\bar{p}\omega^2)^2} \right) + \\
& \left. \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega) \hat{q}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{(\bar{p}\omega^2)^2} \right) \right| \\
= & \left| \frac{1}{N} i(j) i(k) \sum_{\omega} \hat{p}(j-\omega) \hat{p}(\omega-k) \Delta t \left(\frac{6}{\bar{p}} + \frac{8\bar{q}}{(\bar{p}\omega)^2} \right) + \right. \\
& \frac{1}{N} i(j) i \sum_{\omega} \hat{p}(j-\omega) \hat{q}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega} + \frac{8\bar{q}}{\bar{p}^2 \omega^3} \right) + \\
& \frac{1}{N} i(k) i \sum_{\omega} \hat{q}(j-\omega) \hat{p}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega} + \frac{8\bar{q}}{\bar{p}^2 \omega^3} \right) + \\
& \left. \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega) \hat{q}(\omega-k) \Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{(\bar{p}\omega^2)^2} \right) \right|
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \left| \frac{1}{N} i(j) i(k) \sum_{\omega} \hat{p}(j-\omega)\hat{p}(\omega-k)\Delta t \left(\frac{6}{\bar{p}} + \frac{8\bar{q}}{\bar{p}^2} \right) + \right. \\
& \frac{1}{N} i(j) i \sum_{\omega} \hat{p}(j-\omega)\hat{q}(\omega-k)\Delta t \left(\frac{6}{\bar{p}\omega} + \frac{8\bar{q}}{\bar{p}^2\omega} \right) + \\
& \frac{1}{N} i(k) i \sum_{\omega} \hat{q}(j-\omega)\hat{p}(\omega-k)\Delta t \left(\frac{6}{\bar{p}\omega} + \frac{8\bar{q}}{\bar{p}^2\omega} \right) + \\
& \left. \frac{1}{N} \sum_{\omega} \hat{q}(j-\omega)\hat{q}(\omega-k)\Delta t \left(\frac{6}{\bar{p}\omega^2} + \frac{8\bar{q}}{\bar{p}^2\omega^2} \right) \right| .
\end{aligned}
$$

From these bounds, we obtain

$$\begin{aligned}
\|G_{12}\|_\infty &\le \max_{1\le j\le N}\sum_{k=1}^{N}\left|\left(A_{jk}+B_{jk}+C_{jk}+D_{jk}\right)(\bar{p}j^2+\bar{q})^{-1/2}\right| \\
&\le \max_{1\le j\le N}\sum_{k=1}^{N}\left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\bar{q}}{\bar{p}}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{\sqrt{N}}\hat{q}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\bar{q}}{\bar{p}}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{\sqrt{N}}jk\hat{p}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{\sqrt{N}}\hat{q}(j-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{N}i(j)i(k)\sum_{\omega}\hat{p}(j-\omega)\hat{p}(\omega-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{N}i(j)i\sum_{\omega}\hat{p}(j-\omega)\hat{q}(\omega-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{N}i(k)i\sum_{\omega}\hat{q}(j-\omega)\hat{p}(\omega-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\right|+ \\
&\quad \sum_{k=1}^{N}\left|\frac{1}{N}\sum_{\omega}\hat{q}(j-\omega)\hat{q}(\omega-k)(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega^2}+\frac{8\bar{q}}{\bar{p}^2\omega^2}\right)\right| \\
&\le \max_{1\le j\le N}\frac{1}{\sqrt{N}}jN^{1/2}\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=1}^{N}|k|+ \\
&\quad \frac{1}{\sqrt{N}}N^{1/2}\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=1}^{N}|1|+ \\
&\quad \frac{1}{\sqrt{N}}jN^{1/2}\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=1}^{N}|k|+ \\
&\quad \frac{1}{\sqrt{N}}N^{1/2}\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=1}^{N}|1|+ \\
&\quad \frac{1}{N}jN^{1/2}\|\tilde{p}\|_\infty N^{1/2}\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)\sum_{k=1}^{N}|k|+ \\
&\quad \frac{1}{N}jN^{1/2}\|\tilde{p}\|_\infty N^{1/2}\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\sum_{k=1}^{N}\left|\sum_{\omega}\omega\right|+ \\
&\quad \frac{1}{N}N^{1/2}\|\tilde{q}\|_\infty N^{1/2}\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\sum_{k=1}^{N}\left|k\sum_{\omega}\omega\right|+ \\
&\quad \frac{1}{N}N^{1/2}\|\tilde{q}\|_\infty N^{1/2}\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\bar{p}\omega^2}+\frac{8\bar{q}}{\bar{p}^2\omega^2}\right)\sum_{k=1}^{N}\left|\sum_{\omega}\right|
\end{aligned}$$

$$
\begin{aligned}
\leq \quad & \max_{1\leq j\leq N} j\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\overline{q}}{\overline{p}}\right)\sum_{k=j-\omega_{max},k\neq j}^{j+\omega_{max}}|k|+ \\
& \|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\overline{q}}{\overline{p}}\right)2\omega_{max}+ \\
& j\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=j-\omega_{max},k\neq j}^{j+\omega_{max}}|k|+ \\
& |\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)2\omega_{max}+ \\
& j\|\tilde{p}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k|+ \\
& j\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)2\omega_{max}\sum_{\omega=j-\omega_{max}}^{j+\omega_{max}}|\omega|+ \\
& \|\tilde{q}\|_\infty\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\sum_{\omega=j-\omega_{max}}^{j+\omega_{max}}|\omega|\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k|+ \\
& \|\tilde{q}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)4\omega_{max}^2 \\
\leq \quad & \max_{1\leq j\leq N} j\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\overline{q}}{\overline{p}}\right)\sum_{k=j-\omega_{max},k\neq j}^{j+\omega_{max}}|k|+ \\
& j\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)\sum_{k=j-\omega_{max},k\neq j}^{j+\omega_{max}}|k|+ \\
& \|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(3+\frac{4\overline{q}}{\overline{p}}\right)2\omega_{max}+ \\
& |\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(4+\frac{4\bar{q}}{\bar{p}}\right)2\omega_{max}+ \\
& j\|\tilde{p}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k|+ \\
& j\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)2\omega_{max}\sum_{\omega=j-\omega_{max}}^{j+\omega_{max}}|\omega|+ \\
& \|\tilde{q}\|_\infty\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\sum_{\omega=j-\omega_{max}}^{j+\omega_{max}}|\omega|\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k|+ \\
& \|\tilde{q}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega^2}+\frac{8\bar{q}}{\bar{p}^2\omega^2}\right)4\omega_{max}^2
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \max_{1\leq j\leq N} j\|\tilde{p}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(7+\frac{8\overline{q}}{\overline{p}}\right)\sum_{k=j-\omega_{max},k\neq j}^{j+\omega_{max}}|k| + \\
& \|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(7+\frac{8\overline{q}}{\overline{p}}\right)2\omega_{max} + \\
& j\|\tilde{p}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}}+\frac{8\bar{q}}{\bar{p}^2}\right)\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k| + \\
& \|\tilde{p}\|_\infty\|\tilde{q}\|_\infty(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega}+\frac{8\bar{q}}{\bar{p}^2\omega}\right)\sum_{\omega=j-\omega_{max}}^{j+\omega_{max}}|\omega|\left(2j\omega_{max}+\sum_{k=\omega-\omega_{max}k\neq\omega}^{\omega+\omega_{max}}|k|\right) \\
& \|\tilde{q}\|_\infty^2(\bar{p}j^2+\bar{q})^{-1/2}\Delta t\left(\frac{6}{\overline{p}\omega^2}+\frac{8\bar{q}}{\bar{p}^2\omega^2}\right)4\omega_{max}^2.
\end{aligned}
$$

Working out these summations as done in the proof of Lemma 3.2.1, we get

$$
\begin{aligned}
\|G_{12}\|_\infty \quad \leq \quad & C_{12,p}\|\tilde{p}\|_\infty\Delta t N + C_{12,q}\|\tilde{q}\|_\infty\Delta t + C_{12,p^2}\|\tilde{p}\|_\infty^2\Delta t N + \\
& C_{12,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty\Delta t + C_{12,q^2}\|\tilde{q}\|_\infty^2\Delta t.
\end{aligned}
$$

□

Using the same approach, the matrix G_{21} in (3.4) satisfies a similar bound.

3.3 Putting It All Together

We prove that the first-order KSS method applied to (1.1), (1.2), (1.3) is unstable.

Theorem 3.3.1. *Assume* $\hat{p}(\omega)=0$ *and* $\hat{q}(\omega)=0$ *for* $|\omega|\geq\omega_{\max}$. *Then the solution operator* $\tilde{P}$ *satisfies*

$$
\|\tilde{P}\|_C \leq 1+(C_p\|\tilde{p}\|_\infty N + C_q\|\tilde{q}\|_\infty)\Delta t. \tag{3.16}
$$

Proof. Because $\|G\|_\infty$ is the maximum row sum,

$$
\left\|\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}\right\|_\infty \leq \max\{\|G_{11}\|_\infty+\|G_{12}\|_\infty, \|G_{21}\|_\infty+\|G_{22}\|_\infty\}. \tag{3.17}
$$

Then the whole operator is bounded by $\|\tilde{P}\|_C = \|B\|_2 \leq \sqrt{\|G\|_\infty} \leq \sqrt{\|G_{11}\|_\infty + \|G_{12}\|_\infty}$. Using the results from Lemma 3.2.1 and Lemma 3.2.2, we have

$$
\begin{aligned}
\|G\|_\infty \quad \leq \quad & 1+\Delta t N\left(C_{12,p}\|\tilde{p}\|_\infty + C_{12,p^2}\|\tilde{p}\|_\infty^2\right) + \Delta t^2 N^2\left(C_{11,p}\|\tilde{p}\|_\infty + C_{11,p^2}\|\tilde{p}\|_\infty^2\right) + \\
& \Delta t\left(C_{12,q}\|\tilde{q}\|_\infty + C_{12,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty + C_{12,q^2}\|\tilde{q}\|_\infty^2\right) + \\
& \Delta t^2\left(C_{11,q}\|\tilde{q}\|_\infty + C_{11,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty + C_{11,q^2}\|\tilde{q}\|_\infty^2\right).
\end{aligned} \tag{3.18}
$$

Let

$$R = N\left(C_{12,p}\|\tilde{p}\|_\infty + C_{12,p^2}\|\tilde{p}\|_\infty^2\right) + C_{12,q}\|\tilde{q}\|_\infty + C_{12,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty + C_{12,q^2}\|\tilde{q}\|_\infty^2$$

and

$$S = N^2\left(C_{11,p}\|\tilde{p}\|_\infty + C_{11,p^2}\|\tilde{p}\|_\infty^2\right) + C_{11,q}\|\tilde{q}\|_\infty + C_{11,pq}\|\tilde{p}\|_\infty\|\tilde{q}\|_\infty + C_{11,q^2}\|\tilde{q}\|_\infty^2.$$

Let $T = \max\{S^{1/2}, R/2\}$. Then $T \geq R/2$, so $2T \geq R$. We have

$$\|\tilde{P}\|_C \leq \sqrt{\|G\|_\infty} \leq 1 + T\Delta t \leq 1 + (C_p\|\tilde{p}\|_\infty N + C_q\|\tilde{q}\|_\infty)\Delta t$$

from which the result follows. □

Corollary 3.3.2. *Assume the leading coefficient $p(x)$ is constant. Then,*

$$\|\tilde{P}\|_C \leq 1 + (C_q\|\tilde{q}\|_\infty)\Delta t.$$

Proof. The leading coefficient $p(x)$ is constant which means $\tilde{p}(x) = 0$. Then $\tilde{p} = p - \bar{p} = 0$, where $\bar{p}$ is the mean of p. Then from (3.18), we now have

$$\|G\|_\infty \leq 1 + \Delta t\left(C_{12,q}\|\tilde{q}\|_\infty + C_{12,q^2}\|\tilde{q}\|_\infty^2\right) + \Delta t^2\left(C_{11,q}\|\tilde{q}\|_\infty + C_{11,q^2}\|\tilde{q}\|_\infty^2\right).$$

Let $R = C_{12,q}\|\tilde{q}\|_\infty + C_{12,q^2}\|\tilde{q}\|_\infty^2$, $S = C_{11,q}\|\tilde{q}\|_\infty + C_{11,q^2}\|\tilde{q}\|_\infty^2$, and $T = \max\{S^{1/2}, R/2\}$. Then $T \geq R/2$, so $2T \geq R$. We have

$$\|B\|_2 \leq \sqrt{\|G\|_\infty} \leq 1 + T\Delta t \leq 1 + (C_q\|\tilde{q}\|_\infty)\Delta t.$$

Therefore, a first-order KSS method applied to (1.1), (1.2), (1.3) under the assumption that the leading coefficient is constant is unconditionally stable. □

Chapter 4

CONCLUSION

We have proved that a first-order KSS method applied to the variable coefficient wave equation with periodic boundary conditions is not stable with respect to the norm induced by averaging the coefficients We have also proved that the same KSS method applied to the wave equation with periodic boundary conditions is unconditionally stable in the case of a constant leading coefficient.. This is the first result proving unconditional stability for a KSS method with prescribed nodes applied to the wave equation, as opposed to nodes of Gauss quadrature rules.

Ideas for future work include: 1) to prove the same result holds if coefficients are not bandlimited but sufficiently smooth, 2) complete the convergence analysis for the constant leading coefficient case to include consistency and convergence results, 3) generalize the analysis to higher-order KSS methods or higher space dimensions with other boundary conditions, 4) modify the KSS method to ensure stability, and 5) complete a stability analysis of a second-order KSS method applied to the same variable coefficient wave equation using the L-norm.

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