The Axiom of Infinity and The Natural Numbers

Bernd Schröder
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3. In fact, the superstructure over the empty set is a model that satisfies all the axioms so far and which does not contain any infinite sets.
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1. The axioms that we have introduced so far provide for a rich theory.
2. But they do not guarantee the existence of infinite sets.
3. In fact, the superstructure over the empty set is a model that satisfies all the axioms so far and which does not contain any infinite sets. (Remember that the superstructure itself is not a set in the model.)
The Axiom of Infinity

There is a set $I$ that contains $\emptyset$ as an element, and for each $a \in I$ the set $a \cup \{a\}$ is also in $I$. In some ways this axiom says we can "cut across" the different levels of a superstructure and still obtain a set. The superstructure over $I$ is a model that satisfies all axioms introduced so far.
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The superstructure over $I$ is a model that satisfies all axioms introduced so far.
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2. For each $n \in \mathbb{N}$, there is a corresponding element $n' \in \mathbb{N}$, called the successor of $n$. 
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4. Principle of Induction. If $S \subseteq \mathbb{N}$ is such that 1 $\in S$ and for each $n \in S$ we also have $n' \in S$, then $S = \mathbb{N}$.
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The above properties are also called the Peano Axioms for the natural numbers.
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Proof. (Defining $\mathbb{N}$.)

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\[ 1 := \{\emptyset\} = \emptyset \cup \{\emptyset\} \in I. \]

For each $n \in I$, let $n' := n \cup \{n\}$. 
**Proof.** \textit{(Defining \(\mathbb{N}\).)}

Let \(I\) be the set from the Axiom of Infinity. Let
\[1 := \{\emptyset\} = \emptyset \cup \{\emptyset\} \in I.\]
For each \(n \in I\), let \(n' := n \cup \{n\}\).

Call a subset \(S \subseteq I\) a **successor set** iff \(\emptyset \not\in S\), \(1 \in S\) and for all \(n \in S\) we have that \(n' \in S\).
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Moreover, all successor sets are subsets of \( I \).
**Proof.** (Defining \( \mathbb{N} \).)

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Call a subset \( S \subseteq I \) a **successor set** iff \( \emptyset \not\in S \), \( 1 \in S \) and for all \( n \in S \) we have that \( n' \in S \). Then \( I \setminus \{\emptyset\} \) is a successor set.

Moreover, all successor sets are subsets of \( I \). Define \( \mathbb{N} := \bigcap \mathcal{S} \) to be the intersection of the set \( \mathcal{S} \) of all successor sets.
Proof of part 1.

There is a special element in \( \mathbb{N} \), which we denote by 1. Every successor set contains 1. Therefore 1 \( \in \bigcap S = \mathbb{N} \), as was to be proved.
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Proof of part 2. (For each \( n \in \mathbb{N} \), there is a corresponding element \( n' \in \mathbb{N} \), called the successor of \( n \).)

Let \( n \in \mathbb{N} \). Because \( n \in \mathbb{N} = \bigcap \mathcal{S} \), we conclude that \( n \in S \) for all \( S \in \mathcal{S} \).
Proof of part 2. (For each $n \in \mathbb{N}$, there is a corresponding element $n' \in \mathbb{N}$, called the successor of $n$.)

Let $n \in \mathbb{N}$. Because $n \in \mathbb{N} = \bigcap S$, we conclude that $n \in S$ for all $S \in \mathcal{S}$. By definition of successor sets, $n' = n \cup \{n\} \in S$ for all $S \in \mathcal{S}$. 
Proof of part 2. *(For each \( n \in \mathbb{N} \), there is a corresponding element \( n' \in \mathbb{N} \), called the **successor** of \( n \)).*

Let \( n \in \mathbb{N} \). Because \( n \in \mathbb{N} = \bigcap \mathcal{S} \), we conclude that \( n \in S \) for all \( S \in \mathcal{S} \). By definition of successor sets, \( n' = n \cup \{n\} \in S \) for all \( S \in \mathcal{S} \). Hence \( n' \in \bigcap \mathcal{S} = \mathbb{N} \), as was to be proved.
Proof of part 3.

Suppose for a contradiction that $1$ was the successor $1 = x'$ of an $x \in \mathbb{N}$. Then $\{0\} = 1 = x' = x \cup \{x\}$. This implies $x = \{0\}$, but $\{0\} \not\in \mathbb{N}$. We have arrived at a contradiction.
Proof of part 3. *(The element 1 is not the successor of any natural number.)*
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Proof of part 3. *(The element 1 is not the successor of any natural number.)*

Suppose for a contradiction that 1 was the successor $1 = x'$ of an $x \in \mathbb{N}$. Then $\{\emptyset\} = 1 = x' = x \cup \{x\}$. This implies $x = \emptyset$. 
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Suppose for a contradiction that 1 was the successor $1 = x'$ of an $x \in \mathbb{N}$. Then $\{\emptyset\} = 1 = x' = x \cup \{x\}$. This implies $x = \emptyset$, but $\emptyset \notin \mathbb{N}$. 
**Proof of part 3.** *(The element 1 is not the successor of any natural number.)*

Suppose for a contradiction that 1 was the successor \(1 = x'\) of an \(x \in \mathbb{N}\). Then \(\{\emptyset\} = 1 = x' = x \cup \{x\}\). This implies \(x = \emptyset\), but \(\emptyset \not\in \mathbb{N}\). We have arrived at a contradiction.
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Let $S \subseteq \mathbb{N}$ be so that $1 \in S$ and so that for every $n \in S$ we have that $n' \in S$. Because $S$ is a successor set, by definition of $\mathbb{N}$ we conclude $\mathbb{N} \subseteq S$. 
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Proof of part 5.

We first use part 4 to prove that every element of \( n \in \mathbb{N} \) is a subset of \( n \).

Let \( S = \{ n \in \mathbb{N} : \forall m \in n : m \subseteq n \} \).

Trivially, \( \{0\} \in S \), that is, \( 1 \in S \).

For \( n \in S \) we have \( n' = n \cup \{n\} \).

Hence, if \( m \in n' \), then \( m = n \subseteq n' \) or \( m \in n \), which means \( m \subseteq n \subseteq n' \).

Now let \( m, n \in \mathbb{N} \) with \( m' = n' \) be arbitrary but fixed.

Then \( m \cup \{m\} = m' = n' = n \cup \{n\} \).

Suppose for a contradiction that \( m \neq n \).

Then \( \{n\} \neq \{m\} \), which implies \( n \in m \) and \( m \in n \).

By the above, \( n \subseteq m \) and \( m \subseteq n \), that is, \( m = n \), contradiction.
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Now let \( m, n \in \mathbb{N} \) with \( m' = n' \) be arbitrary but fixed.
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Now let \(m, n \in \mathbb{N}\) with \(m' = n'\) be arbitrary but fixed. Then \(m \cup \{m\} = m' = n'\).
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Now let $m, n \in \mathbb{N}$ with $m' = n'$ be arbitrary but fixed. Then $m \cup \{m\} = m' = n' = n \cup \{n\}$. Suppose for a contradiction that $m \neq n$. 
Proof of part 5. \((\text{For all } m, n \in \mathbb{N} \text{ if } m' = n', \text{ then } m = n.)\)

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Now let \(m, n \in \mathbb{N}\) with \(m' = n'\) be arbitrary but fixed. Then \(m \cup \{m\} = m' = n' = n \cup \{n\}\). Suppose for a contradiction that \(m \neq n\). Then \(\{n\} \neq \{m\}\).
Proof of part 5. \((For\ all\ m, n \in \mathbb{N}\ if\ m' = n',\ then\ m = n.\)\)

We first use part 4 to prove that every element of \(n \in \mathbb{N}\) is a subset of \(n\). Let \(S := \{n \in \mathbb{N} : \forall m \in n : m \subseteq n\}\). Trivially, \(\{\emptyset\} \in S\), that is, \(1 \in S\). For \(n \in S\) we have \(n' = n \cup \{n\}\). Hence, if \(m \in n'\), then \(m = n \subseteq n'\) or \(m \in n\), which means \(m \subseteq n \subseteq n'\).

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Now let \( m, n \in \mathbb{N} \) with \( m' = n' \) be arbitrary but fixed. Then \( m \cup \{m\} = m' = n' = n \cup \{n\} \). Suppose for a contradiction that \( m \neq n \). Then \( \{n\} \neq \{m\} \), which implies \( n \in m \) and \( m \in n \). By the above, \( n \subseteq m \).
Proof of part 5. (For all $m,n \in \mathbb{N}$ if $m' = n'$, then $m = n$.)

We first use part 4 to prove that every element of $n \in \mathbb{N}$ is a subset of $n$. Let $S := \{ n \in \mathbb{N} : \forall m \in n : m \subseteq n \}$. Trivially, $\{\emptyset\} \in S$, that is, $1 \in S$. For $n \in S$ we have $n' = n \cup \{n\}$. Hence, if $m \in n'$, then $m = n \subseteq n'$ or $m \in n$, which means $m \subseteq n \subseteq n'$.

Now let $m,n \in \mathbb{N}$ with $m' = n'$ be arbitrary but fixed. Then $m \cup \{m\} = m' = n' = n \cup \{n\}$. Suppose for a contradiction that $m \neq n$. Then $\{n\} \neq \{m\}$, which implies $n \in m$ and $m \in n$. By the above, $n \subseteq m$ and $m \subseteq n$.
Proof of part 5. *(For all \(m, n \in \mathbb{N}\) if \(m' = n'\), then \(m = n\).)*

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**Proof of part 5.** *(For all \( m, n \in \mathbb{N} \) if \( m' = n' \), then \( m = n \)).*

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9. For example, to start, the engine must turn over. The handcrank from the really old movies has been replaced with an electric motor that cranks as we turn the key. Knowing that we need the engine to turn over is helpful when starting a car with electrical problems.