The Axiom of Choice

Bernd Schröder
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2. ... and by the desire to have “standard sizes” for infinite sets.
Why Do We Need More Axioms?

1. The final three axioms we discuss could be motivated by the desire to “count past infinity” ...
2. ... and by the desire to have “standard sizes” for infinite sets.
3. Other than that, the Axiom of Choice, in its “Zorn’s Lemma” incarnation is used every so often throughout mathematics.
Axiom.
**Axiom. The Axiom of Choice.** Let \( \{A_i\}_{i \in I} \) be an indexed family of sets.
**Axiom. The Axiom of Choice.** Let \( \{A_i\}_{i \in I} \) be an indexed family of sets. Then there is a function \( f : I \rightarrow \bigcup_{i \in I} A_i \) so that \( f(i) \in A_i \) for all \( i \in I \).
Axiom. The Axiom of Choice. Let $\{A_i\}_{i \in I}$ be an indexed family of sets. Then there is a function $f : I \rightarrow \bigcup_{i \in I} A_i$ so that $f(i) \in A_i$ for all $i \in I$. The function is also called a choice function.
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Definition.
**Axiom. The Axiom of Choice.** Let \( \{A_i\}_{i \in I} \) be an indexed family of sets. Then there is a function \( f : I \rightarrow \bigcup_{i \in I} A_i \) so that \( f(i) \in A_i \) for all \( i \in I \). The function is also called a **choice function**.

**Definition.** Let \( \{A_i\}_{i \in I} \) be a family of sets.
Axiom. The Axiom of Choice. Let \( \{A_i\}_{i \in I} \) be an indexed family of sets. Then there is a function \( f : I \to \bigcup_{i \in I} A_i \) so that \( f(i) \in A_i \) for all \( i \in I \). The function is also called a choice function.

Definition. Let \( \{A_i\}_{i \in I} \) be a family of sets. The product \( \prod_{i \in I} A_i \) of the \( A_i \) is defined as the set of all functions \( f : I \to \bigcup_{i \in I} A_i \) with \( f(i) \in A_i \) for all \( i \in I \).
Theorem.
Theorem. *Intersection and union are completely distributive.*
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Let \( \{J_i\}_{i \in I} \) be a family of index sets and let \( \{C_{ij}\}_{i \in I, j \in J_i} \) be a family of sets.
Theorem. Intersection and union are completely distributive.
Let \( \{J_i\}_{i \in I} \) be a family of index sets and let \( \{C_{ij}\}_{i \in I, j \in J_i} \) be a family of sets. Then the following hold.
Theorem. Intersection and union are completely distributive. Let \( \{J_i\}_{i \in I} \) be a family of index sets and let \( \{C_{ij}\}_{i \in I, j \in J_i} \) be a family of sets. Then the following hold.

1. \( \bigcap_{i \in I} \bigcup_{j \in J_i} C_{ij} = \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)} \).
**Theorem.** *Intersection and union are completely distributive.*

Let \( \{J_i\}_{i \in I} \) be a family of index sets and let \( \{C_{ij}\}_{i \in I, j \in J_i} \) be a family of sets. Then the following hold.

1. \[
\bigcap_{i \in I} \bigcup_{j \in J_i} C_{ij} = \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}.
\]

2. \[
\bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} = \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)}.
\]
Proof (part 1).
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Let \( x \in \bigcap \bigcup_{i \in I} C_{ij} \).
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Let $x \in \bigcap \bigcup \limits_{i \in I} \bigcup \limits_{j \in J_i} C_{ij}$. Then for every $i \in I$ there is a $j_i$ with $x \in C_{ij_i}$. 

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Proof (part 1).

Let \( x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij} \). Then for every \( i \in I \) there is a \( j_i \) with \( x \in C_{ij_i} \).

For each \( i \in I \) set \( g(i) := j_i \).
Proof (part 1).

Let $x \in \bigcap \bigcup_{i \in I \atop j \in J_i} C_{ij}$. Then for every $i \in I$ there is a $j_i$ with $x \in C_{ij_i}$.

For each $i \in I$ set $g(i) := j_i$. Then

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For each \( i \in I \) set \( g(i) := j_i \). Then

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x \in \bigcap_{i \in I} C_{ig(i)}
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Proof (part 1).
Let $x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij}$. Then for every $i \in I$ there is a $j_i$ with $x \in C_{ij_i}$.

For each $i \in I$ set $g(i) := j_i$. Then

$x \in \bigcap_{i \in I} C_{ig(i)} \subseteq \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}$. 

Conversely, let $x \in \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}$.
Then there is a choice function $f \in \prod_{i \in I} J_i$ so that $x \in C_{if(i)}$ for all $i \in I$.

But then $x \in \bigcup_{j \in J_i} C_{ij}$ for every $i \in I$.

Hence $x \in \bigcap_{i \in I} \bigcup_{j \in J_i} C_{ij}$. 

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Proof (part 1).

Let \( x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij} \). Then for every \( i \in I \) there is a \( j_i \) with \( x \in C_{ij_i} \).

For each \( i \in I \) set \( g(i) := j_i \). Then
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x \in \bigcap_{i \in I} C_{ig(i)} \subseteq \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}.
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Conversely, let
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x \in \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}.
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For each \( i \in I \) set \( g(i) := j_i \). Then

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\bigcap_{i \in I} C_{ig(i)} \subseteq \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}.
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Conversely, let \( x \in \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)} \). Then there is a choice function \( f \in \prod_{i \in I} J_i \) so that \( x \in C_{if(i)} \) for all \( i \in I \).
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Let \( x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij} \). Then for every \( i \in I \) there is a \( j_i \) with \( x \in C_{ij_i} \).

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\bar{x} \in \bigcap_{i \in I} C_{ig(i)} \subseteq \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)}.
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Conversely, let \( x \in \bigcup_{f \in \prod_{i \in I} J_i} \bigcap_{i \in I} C_{if(i)} \). Then there is a choice function \( f \in \prod_{i \in I} J_i \) so that \( x \in C_{if(i)} \) for all \( i \in I \). But then \( x \in \bigcup_{j \in J_i} C_{ij} \) for every \( i \in I \).
**Proof (part 1).**

Let \( x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij} \). Then for every \( i \in I \) there is a \( j_i \) with \( x \in C_{ij_i} \).

For each \( i \in I \) set \( g(i) := j_i \). Then

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Hence \( x \in \bigcap \bigcup_{i \in I, j \in J_i} C_{ij} \).
Proof (part 2).

Let $x \in \bigcup_{i \in I} \bigcap_{j \in J} i C_{ij}$.

Then there is an $i_0 \in I$ so that $x \in \bigcap_{j \in J} i_0 C_{i_0 j}$.

For every choice function $f \in \prod_{i \in I} J$ we have $x \in \bigcap_{j \in J} i_0 C_{i_0 j} \subseteq C_{i_0 f(i_0)} \subseteq \bigcup_{i \in I} C_{if(i_0)}$.

Therefore $x \in \bigcap_{f \in \prod_{i \in I} J} \bigcup_{i \in I} C_{if(i_0)}$.

For the reverse inclusion, let $x \not\in \bigcup_{i \in I} \bigcap_{j \in J} i C_{ij}$.

Then for every $i \in I$ there is a $j_i \in J$ so that $x \not\in C_{ij_i}$.

Define $g \in \prod_{i \in I} J$ by $g(i) = j_i$.

Then $x \not\in \bigcup_{i \in I} C_{ig(i)}$ and hence $x \not\in \bigcap_{f \in \prod_{i \in I} J} \bigcup_{i \in I} C_{if(i)}$.
Proof (part 2).

Let \( x \in \bigcup \bigcap_{i \in I} \bigcap_{j \in J_i} C_{ij} \).
Proof (part 2).

Let $x \in \bigcup \bigcap_{i \in I} \bigcap_{j \in J_i} C_{ij}$. Then there is an $i_0 \in I$ so that $x \in \bigcap_{j \in J_{i_0}} C_{i_0j}$.
Proof (part 2).

Let \( x \in \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} \). Then there is an \( i_0 \in I \) so that \( x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \).

For every choice function \( f \in \prod_{i \in I} J_i \) we have

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For every choice function \( f \in \prod_{i \in I} J_i \) we have

\[
x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)}
\]
Proof (part 2).

Let \( x \in \bigcup \bigcap_{i \in I, j \in J_i} C_{ij} \). Then there is an \( i_0 \in I \) so that \( x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \).

For every choice function \( f \in \prod_{i \in I} J_i \) we have

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x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)} \subseteq \bigcup_{i \in I} C_{if(i)}.
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Therefore \( x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)} \).
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Therefore \( x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)} \).

For the reverse inclusion, let \( x \not\in \bigcup \bigcap_{i \in I} C_{ij} \).
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Let \( x \in \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} \). Then there is an \( i_0 \in I \) so that \( x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \).

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Therefore \( x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)} \).

For the reverse inclusion, let \( x \notin \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} \). Then for every \( i \in I \) there is a \( j_i \in J_i \) so that \( x \notin C_{ij_i} \).
Proof (part 2).

Let $x \in \bigcup \bigcap_{i \in I} C_{ij}$. Then there is an $i_0 \in I$ so that $x \in \bigcap_{j \in J_{i_0}} C_{i_0j}$.

For every choice function $f \in \prod_{i \in I} J_i$ we have

$$x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)} \subseteq \bigcup_{i \in I} C_{if(i)}.$$  Therefore $x \in \bigcap_{f \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)}$.

For the reverse inclusion, let $x \not\in \bigcup \bigcap_{i \in I} C_{ij}$. Then for every $i \in I$ there is a $j_i \in J_i$ so that $x \not\in C_{ij_i}$. Define $g \in \prod_{i \in I} J_i$ by $g(i) := j_i$. 
Proof (part 2).

Let \( x \in \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij}. \) Then there is an \( i_0 \in I \) so that \( x \in \bigcap_{j \in J_{i_0}} C_{i_0j}. \)

For every choice function \( f \in \prod_{i \in I} J_i \) we have

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x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)} \subseteq \bigcup_{i \in I} C_{if(i)}.\]

Therefore \( x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)}. \)

For the reverse inclusion, let \( x \notin \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij}. \) Then for every \( i \in I \)

there is a \( j_i \in J_i \) so that \( x \notin C_{ij_i}. \) Define \( g \in \prod_{i \in I} J_i \) by \( g(i) := j_i. \)

Then \( x \notin \bigcup_{i \in I} C_{ig(i)} \)
Proof (part 2).

Let $x \in \bigcup \bigcap_{i \in I, j \in J_i} C_{ij}$. Then there is an $i_0 \in I$ so that $x \in \bigcap_{j \in J_{i_0}} C_{i_0j}$.

For every choice function $f \in \prod_{i \in I} J_i$ we have

$$x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)} \subseteq \bigcup_{i \in I} C_{if(i)}.$$ Therefore $x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)}$.

For the reverse inclusion, let $x \notin \bigcup \bigcap_{i \in I, j \in J_i} C_{ij}$. Then for every $i \in I$ there is a $j_i \in J_i$ so that $x \notin C_{ij_i}$. Define $g \in \prod_{i \in I} J_i$ by $g(i) := j_i$.

Then $x \notin \bigcup_{i \in I} C_{ig(i)}$ and hence $x \notin \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{if(i)}$. 

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Proof (part 2).

Let \( x \in \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} \). Then there is an \( i_0 \in I \) so that \( x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \).

For every choice function \( f \in \prod_{i \in I} J_i \) we have

\[
x \in \bigcap_{j \in J_{i_0}} C_{i_0j} \subseteq C_{i_0f(i_0)} \subseteq \bigcup_{i \in I} C_{i_f(i)}. \text{ Therefore } x \in \bigcap_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{i_f(i)}.
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For the reverse inclusion, let \( x \notin \bigcup_{i \in I} \bigcap_{j \in J_i} C_{ij} \). Then for every \( i \in I \)

there is a \( j_i \in J_i \) so that \( x \notin C_{ij_i} \). Define \( g \in \prod_{i \in I} J_i \) by \( g(i) := j_i \).

Then \( x \notin \bigcup_{i \in I} C_{ig(i)} \) and hence \( x \notin \bigcup_{f \in \prod_{i \in I} J_i} \bigcup_{i \in I} C_{i_f(i)} \). \( \square \)
Definition.
**Definition.** Let $X$ be an ordered set.
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**Lemma.** Let $X$ be a set, and let $Z \subseteq P(X)$ be a nonempty set of subsets of $X$, ordered by set containment $\subseteq$ and with the following properties.

1. For every set $C \in Z$ we have that every subset of $C$ is an element of $Z$.
2. For every chain (with respect to set containment) $C \subseteq Z$, the union $\bigcup C$ of $C$ is an element of $Z$.

Then $Z$ has a maximal element (with respect to set containment).
**Definition.** Let $X$ be an ordered set. A totally ordered subset $C$ of $X$ is also called a **chain**. An element $m \in X$ so that for all $x \in X$ we have that $m \leq x$ implies $m = x$ is called a **maximal element** of $X$. 
**Definition.** Let $X$ be an ordered set. A totally ordered subset $C$ of $X$ is also called a **chain**. An element $m \in X$ so that for all $x \in X$ we have that $m \leq x$ implies $m = x$ is called a **maximal element** of $X$.

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**Definition.** Let $X$ be an ordered set. A totally ordered subset $C$ of $X$ is also called a **chain**. An element $m \in X$ so that for all $x \in X$ we have that $m \leq x$ implies $m = x$ is called a **maximal element** of $X$.

**Lemma.** Let $X$ be a set, and let $Z \subseteq \mathcal{P}(X)$ be a nonempty set of subsets of $X$, ordered by set containment $\subseteq$ and with the following properties.

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**Proof (initial setup).** Consider the indexed family \( \{ i \}_{i \in \mathcal{P}(X) \setminus \{ \emptyset \}} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : \mathcal{P}(X) \setminus \{ \emptyset \} \rightarrow X \).
Proof (initial setup). Consider the indexed family \( \{ i \}_{i \in \mathcal{P}(X) \setminus \{ \emptyset \}} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : \mathcal{P}(X) \setminus \{ \emptyset \} \to X \) so that \( f(A) \in A \) holds for all \( A \in \mathcal{P}(X) \setminus \{ \emptyset \} \).
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For each \( C \in Z \), define the set \( E_C := \{x \in X \setminus C : C \cup \{x\} \in Z\} \)
Proof (initial setup). Consider the indexed family \( \{ i \}_{i \in P(X) \setminus \{ \emptyset \}} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : P(X) \setminus \{ \emptyset \} \rightarrow X \) so that \( f(A) \in A \) holds for all \( A \in P(X) \setminus \{ \emptyset \} \).

For each \( C \in Z \), define the set \( E_C := \{ x \in X \setminus C : C \cup \{ x \} \in Z \} \) and let

\[
g(C) := \begin{cases} 
    C \cup \{ f(E_C) \}; & \text{if } E_C \neq \emptyset, \\
    C; & \text{if } E_C = \emptyset.
\end{cases}
\]
**Proof (initial setup).** Consider the indexed family \( \{ i \} \in \mathcal{P}(X) \setminus \{ \emptyset \} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : \mathcal{P}(X) \setminus \{ \emptyset \} \to X \) so that \( f(A) \in A \) holds for all \( A \in \mathcal{P}(X) \setminus \{ \emptyset \} \).

For each \( C \in Z \), define the set \( E_C := \{ x \in X \setminus C : C \cup \{ x \} \in Z \} \) and let

\[
g(C) := \begin{cases} 
  C \cup \{ f(E_C) \}; & \text{if } E_C \neq \emptyset, \\
  C; & \text{if } E_C = \emptyset.
\end{cases}
\]

If \( M \in Z \) satisfies \( g(M) = M \), then there is no element \( x \in X \setminus M \) so that \( M \cup \{ x \} \in Z \).
Proof (initial setup). Consider the indexed family \( \{i\}_{i \in \mathcal{P}(X) \setminus \{\emptyset\}} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \) so that \( f(A) \in A \) holds for all \( A \in \mathcal{P}(X) \setminus \{\emptyset\} \).

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**Proof (initial setup).** Consider the indexed family \( \{i\}_{i \in \mathcal{P}(X) \setminus \{\emptyset\}} \). The union of this family is \( X \). By the Axiom of Choice, there is a choice function \( f : \mathcal{P}(X) \setminus \{\emptyset\} \to X \) so that \( f(A) \in A \) holds for all \( A \in \mathcal{P}(X) \setminus \{\emptyset\} \).

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If \( M \in Z \) satisfies \( g(M) = M \), then there is no element \( x \in X \setminus M \) so that \( M \cup \{x\} \in Z \), which means that \( M \) is maximal in \( Z \). The proof will be done once we find an \( M \in Z \) with \( g(M) = M \).
Proof (towers).
Proof (towers). A subset $T \subseteq Z$ will be called a tower iff

1. 0 ∈ $T$, and
2. If $C \in T$, then $g(C) \in T$, and
3. If $C \subseteq T$ is a chain in $T$, then $\bigcup C \in T$.

The set $Z$ contains at least one tower, because $Z$ itself is a tower. Moreover, the intersection of any set of towers is a tower, too. Let $T_0$ be the intersection of all towers that are contained in $Z$. Then $T_0$ is not empty, because 0 ∈ $T_0$. Call an element $C \in T_0$ comparable iff for all $A \in T_0$ we have $A \subseteq C$ or $C \subseteq A$. First note that, clearly, 0 is a comparable set.
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Proof ($C$ comparable implies $g(C)$ comparable).
Proof \((C \text{ comparable implies } g(C) \text{ comparable})\). Let \(C \in T_0\) be a fixed comparable set.
Proof (C comparable implies g(C) comparable). Let C ∈ T₀ be a fixed comparable set. Consider the set

\[ U := \{ A \in T₀ : A \subseteq C \text{ or } g(C) \subseteq A \} . \]
Proof (\(C\) comparable implies \(g(C)\) comparable). Let \(C \in T_0\) be a fixed comparable set. Consider the set \(U := \{A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A\}\). We will prove that \(U\) is a tower.
Proof (*C* comparable implies *g*(*)C*) comparable). Let *C* ∈ *T*₀ be a fixed comparable set. Consider the set

\[ U := \{ A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A \} \]

We will prove that *U* is a tower, which implies that *U* = *T*₀.
Proof (C comparable implies \( g(C) \) comparable). Let \( C \in T_0 \) be a fixed comparable set. Consider the set
\[
U := \{ A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A \}.
\]
We will prove that \( U \) is a tower, which implies that \( U = T_0 \), which implies that \( g(C) \) is comparable.
Proof (C comparable implies \( g(C) \) comparable). Let \( C \in T_0 \) be a fixed comparable set. Consider the set 
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We will prove that \( U \) is a tower, which implies that \( U = T_0 \), which implies that \( g(C) \) is comparable. Clearly, \( \emptyset \in U \). Now let \( A \in U \).
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Proof (C comparable implies g(C) comparable). Let $C \in T_0$ be a fixed comparable set. Consider the set $U := \{ A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A \}$. We will prove that $U$ is a tower, which implies that $U = T_0$, which implies that $g(C)$ is comparable. Clearly, $\emptyset \in U$. Now let $A \in U$. Because $C$ is comparable, we have $A = C$ or $A \subset C$ or $C \subset A$. In case $A = C$, we have $g(A) = g(C) \supseteq g(C)$, which means $g(A) \in U$. In case $A \subset C$, because $C$ is comparable, $g(A) \subseteq C$ or $C \subset g(A)$. Strict containment $C \subset g(A)$ would mean (by $A \subset C$) that $C$ has at least one more element than $A$ and $g(A)$ has at least one more element than $C$. 
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Proof (C comparable implies $g(C)$ comparable). Let $C \in T_0$ be a fixed comparable set. Consider the set $U := \{ A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A \}$. We will prove that $U$ is a tower, which implies that $U = T_0$, which implies that $g(C)$ is comparable. Clearly, $\emptyset \in U$. Now let $A \in U$. Because $C$ is comparable, we have $A = C$ or $A \subset C$ or $C \subset A$. In case $A = C$, we have $g(A) = g(C) \supseteq g(C)$, which means $g(A) \in U$. In case $A \subset C$, because $C$ is comparable, $g(A) \subseteq C$ or $C \subset g(A)$. Strict containment $C \subset g(A)$ would mean (by $A \subset C$) that $C$ has at least one more element than $A$ and $g(A)$ has at least one more element than $C$, which is impossible. Thus, in case $A \subset C$ we must have $g(A) \subseteq C$. 

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Proof \((C \text{ comparable implies } g(C) \text{ comparable})\). Let \(C \in T_0\) be a fixed comparable set. Consider the set \(U := \{A \in T_0 : A \subseteq C \text{ or } g(C) \subseteq A\}\). We will prove that \(U\) is a tower, which implies that \(U = T_0\), which implies that \(g(C)\) is comparable. Clearly, \(\emptyset \in U\). Now let \(A \in U\). Because \(C\) is comparable, we have \(A = C\) or \(A \subset C\) or \(C \subset A\). In case \(A = C\), we have \(g(A) = g(C) \supseteq g(C)\), which means \(g(A) \in U\). In case \(A \subset C\), because \(C\) is comparable, \(g(A) \subseteq C\) or \(C \subset g(A)\). Strict containment \(C \subset g(A)\) would mean (by \(A \subset C\)) that \(C\) has at least one more element than \(A\) and \(g(A)\) has at least one more element than \(C\), which is impossible. Thus, in case \(A \subset C\) we must have \(g(A) \subseteq C\), which means \(g(A) \in U\).
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We will prove that \(U\) is a tower, which implies that \(U = T_0\), which implies that \(g(C)\) is comparable. Clearly, \(\emptyset \in U\). Now let \(A \in U\). Because \(C\) is comparable, we have \(A = C\) or \(A \subset C\) or \(C \subset A\). In case \(A = C\), we have \(g(A) = g(C) \supseteq g(C)\), which means \(g(A) \in U\). In case \(A \subset C\), because \(C\) is comparable, \(g(A) \subseteq C\) or \(C \subset g(A)\). Strict containment \(C \subset g(A)\) would mean (by \(A \subset C\)) that \(C\) has at least one more element than \(A\) and \(g(A)\) has at least one more element than \(C\), which is impossible. Thus, in case \(A \subset C\) we must have \(g(A) \subseteq C\), which means \(g(A) \in U\). In the last case, \(C \subset A\), we note that \(A \not\subset C\).
Proof (C comparable implies g(C) comparable). Let C ∈ T₀ be a fixed comparable set. Consider the set
U := \{A ∈ T₀ : A ⊆ C or g(C) ⊆ A\}. We will prove that U is a tower, which implies that U = T₀, which implies that g(C) is comparable. Clearly, ∅ ∈ U. Now let A ∈ U. Because C is comparable, we have A = C or A ⊂ C or C ⊂ A. In case A = C, we have g(A) = g(C) ⊇ g(C), which means g(A) ∈ U. In case A ⊂ C, because C is comparable, g(A) ⊆ C or C ⊂ g(A). Strict containment C ⊂ g(A) would mean (by A ⊂ C) that C has at least one more element than A and g(A) has at least one more element than C, which is impossible. Thus, in case A ⊂ C we must have g(A) ⊆ C, which means g(A) ∈ U. In the last case, C ⊂ A, we note that A ⊄ C. Thus, by definition of U, g(C) ⊆ A ⊆ g(A) and g(A) ∈ U.
Proof \((C \text{ comparable implies } g(C) \text{ comparable}, \text{ concl.})\).
Proof \((C \text{ comparable implies } g(C) \text{ comparable, concl.})\).
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Proof (\( C \) comparable implies \( g(C) \) comparable, concl.).

Finally, let \( \mathcal{A} \subseteq U \) be a chain. If \( C \supseteq A \) for all \( A \in \mathcal{A} \), then \( C \supseteq \bigcup \mathcal{A} \) and \( \bigcup \mathcal{A} \in U \).
Proof ($C$ comparable implies $g(C)$ comparable, concl.).

Finally, let $\mathcal{A} \subseteq U$ be a chain. If $C \supseteq A$ for all $A \in \mathcal{A}$, then $C \supseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{A} \in U$. Otherwise, there is an $A \in \mathcal{A}$ so that $C \subset A$. 
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Finally, let \(A \subseteq U\) be a chain. If \(C \supseteq A\) for all \(A \in A\), then \(C \supseteq \bigcup A\) and \(\bigcup A \in U\). Otherwise, there is an \(A \in A\) so that \(C \subset A\). But then \(A \nsubseteq C\).
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Finally, let $A \subseteq U$ be a chain. If $C \supseteq A$ for all $A \in A$, then $C \supseteq \bigcup A$ and $\bigcup A \in U$. Otherwise, there is an $A \in A$ so that $C \subset A$. But then $A \not\subseteq C$, which implies $g(C) \subseteq A \subseteq \bigcup A$ and hence $\bigcup A \in U$. 

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Proof (\(C\) comparable implies \(g(C)\) comparable, concl.).

Finally, let \(\mathcal{A} \subseteq U\) be a chain. If \(C \supseteq A\) for all \(A \in \mathcal{A}\), then \(C \supseteq \bigcup \mathcal{A}\) and \(\bigcup \mathcal{A} \in U\). Otherwise, there is an \(A \in \mathcal{A}\) so that \(C \subset A\). But then \(A \nsubseteq C\), which implies \(g(C) \subseteq A \subseteq \bigcup \mathcal{A}\) and hence \(\bigcup \mathcal{A} \in U\). Thus \(U \subseteq T_0\) is a tower.
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Finally, let $\mathcal{A} \subseteq U$ be a chain. If $C \supseteq A$ for all $A \in \mathcal{A}$, then $C \supseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{A} \in U$. Otherwise, there is an $A \in \mathcal{A}$ so that $C \subset A$. But then $A \not\supseteq C$, which implies $g(C) \subseteq A \subseteq \bigcup \mathcal{A}$ and hence $\bigcup \mathcal{A} \in U$. Thus $U \subseteq T_0$ is a tower. By definition of $T_0$, $T_0 \subseteq U$.
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Finally, let \( A \subseteq U \) be a chain. If \( C \supseteq A \) for all \( A \in A \), then \( C \supseteq \bigcup A \) and \( \bigcup A \in U \). Otherwise, there is an \( A \in A \) so that \( C \subset A \). But then \( A \not\subseteq C \), which implies \( g(C) \subseteq A \subseteq \bigcup A \) and hence \( \bigcup A \in U \). Thus \( U \subseteq T_0 \) is a tower. By definition of \( T_0 \), \( T_0 \subseteq U \) and hence \( U = T_0 \).
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Finally, let $\mathcal{A} \subseteq U$ be a chain. If $C \supseteq A$ for all $A \in \mathcal{A}$, then $C \supseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{A} \in U$. Otherwise, there is an $A \in \mathcal{A}$ so that $C \subset A$. But then $A \not\subseteq C$, which implies $g(C) \subseteq A \subseteq \bigcup \mathcal{A}$ and hence $\bigcup \mathcal{A} \in U$. Thus $U \subseteq T_0$ is a tower. By definition of $T_0$, $T_0 \subseteq U$ and hence $U = T_0$. Thus for all $A \in T_0$ we have $A \subseteq C \subseteq g(C)$ or $g(C) \subseteq A$. 
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Finally, let \( \mathcal{A} \subseteq U \) be a chain. If \( C \supseteq A \) for all \( A \in \mathcal{A} \), then \( C \supseteq \bigcup \mathcal{A} \) and \( \bigcup \mathcal{A} \in U \). Otherwise, there is an \( A \in \mathcal{A} \) so that \( C \subset A \). But then \( A \not\subset C \), which implies \( g(C) \subseteq A \subseteq \bigcup \mathcal{A} \) and hence \( \bigcup \mathcal{A} \in U \). Thus \( U \subseteq T_0 \) is a tower. By definition of \( T_0 \), \( T_0 \subseteq U \) and hence \( U = T_0 \). Thus for all \( A \in T_0 \) we have \( A \subseteq C \subseteq g(C) \) or \( g(C) \subseteq A \). So if \( C \in T_0 \) is comparable, then \( g(C) \) is comparable, too.
Proof (existence of maximal elements).
Proof (existence of maximal elements). Let \( \mathcal{C} \subseteq T_0 \) be a chain of comparable elements and let \( A \in T_0 \).
**Proof (existence of maximal elements).** Let \( \mathcal{C} \subseteq T_0 \) be a chain of comparable elements and let \( A \in T_0 \). If there is a \( C \in \mathcal{C} \) with \( A \subseteq C \), then \( A \subseteq C \subseteq \bigcup \mathcal{C} \).
Proof (existence of maximal elements). Let $\mathcal{C} \subseteq T_0$ be a chain of comparable elements and let $A \in T_0$. If there is a $C \in \mathcal{C}$ with $A \subseteq C$, then $A \subseteq C \subseteq \bigcup \mathcal{C}$. Otherwise for all $C \in \mathcal{C}$ we have $C \subseteq A$, which means $\bigcup \mathcal{C} \subseteq A$. 
Proof (existence of maximal elements). Let \( C \subseteq T_0 \) be a chain of comparable elements and let \( A \in T_0 \). If there is a \( C \in C \) with \( A \subseteq C \), then \( A \subseteq C \subseteq \bigcup C \). Otherwise for all \( C \in C \) we have \( C \subseteq A \), which means \( \bigcup C \subseteq A \). Consequently, if \( C \subseteq T_0 \) is a chain of comparable elements, then the union \( \bigcup C \) is comparable.
Proof (existence of maximal elements). Let \( \mathcal{C} \subseteq T_0 \) be a chain of comparable elements and let \( A \in T_0 \). If there is a \( C \in \mathcal{C} \) with \( A \subseteq C \), then \( A \subseteq C \subseteq \bigcup \mathcal{C} \). Otherwise for all \( C \in \mathcal{C} \) we have \( C \subseteq A \), which means \( \bigcup \mathcal{C} \subseteq A \). Consequently, if \( \mathcal{C} \subseteq T_0 \) is a chain of comparable elements, then the union \( \bigcup \mathcal{C} \) is comparable.

Thus the set of comparable elements in \( T_0 \) is a tower.
Proof (existence of maximal elements). Let $\mathcal{C} \subseteq T_0$ be a chain of comparable elements and let $A \in T_0$. If there is a $C \in \mathcal{C}$ with $A \subseteq C$, then $A \subseteq C \subseteq \bigcup \mathcal{C}$. Otherwise for all $C \in \mathcal{C}$ we have $C \subseteq A$, which means $\bigcup \mathcal{C} \subseteq A$. Consequently, if $\mathcal{C} \subseteq T_0$ is a chain of comparable elements, then the union $\bigcup \mathcal{C}$ is comparable. Thus the set of comparable elements in $T_0$ is a tower. Because $T_0$ is the intersection of all towers, every element of $T_0$ is comparable.
Proof (existence of maximal elements). Let $C \subseteq T_0$ be a chain of comparable elements and let $A \in T_0$. If there is a $C \in C$ with $A \subseteq C$, then $A \subseteq C \subseteq \bigcup C$. Otherwise for all $C \in C$ we have $C \subseteq A$, which means $\bigcup C \subseteq A$. Consequently, if $C \subseteq T_0$ is a chain of comparable elements, then the union $\bigcup C$ is comparable.

Thus the set of comparable elements in $T_0$ is a tower. Because $T_0$ is the intersection of all towers, every element of $T_0$ is comparable. By definition of comparable elements, $T_0$ is a chain.
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$$ g \left( \bigcup T_0 \right) \in T_0 $$
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Proof (existence of maximal elements). Let $\mathcal{C} \subseteq T_0$ be a chain of comparable elements and let $A \in T_0$. If there is a $C \in \mathcal{C}$ with $A \subseteq C$, then $A \subseteq C \subseteq \bigcup \mathcal{C}$. Otherwise for all $C \in \mathcal{C}$ we have $C \subseteq A$, which means $\bigcup \mathcal{C} \subseteq A$. Consequently, if $\mathcal{C} \subseteq T_0$ is a chain of comparable elements, then the union $\bigcup \mathcal{C}$ is comparable.

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The Axiom of Choice
**Proof (existence of maximal elements).** Let $\mathcal{C} \subseteq T_0$ be a chain of comparable elements and let $A \in T_0$. If there is a $C \in \mathcal{C}$ with $A \subseteq C$, then $A \subseteq C \subseteq \bigcup \mathcal{C}$. Otherwise for all $C \in \mathcal{C}$ we have $C \subseteq A$, which means $\bigcup \mathcal{C} \subseteq A$. Consequently, if $\mathcal{C} \subseteq T_0$ is a chain of comparable elements, then the union $\bigcup \mathcal{C}$ is comparable.

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Theorem.
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**Theorem. Zorn’s Lemma.** Let $X$ be a nonempty ordered set so that every chain in $X$ has an upper bound. Then $X$ has a maximal element.
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Proof. Let $Z$ be the set of all chains in $X$, ordered by inclusion. If $C \in Z$, then every subset of $C$ is in $Z$, too.
**Theorem. Zorn’s Lemma.** Let $X$ be a nonempty ordered set so that every chain in $X$ has an upper bound. Then $X$ has a maximal element.

**Proof.** Let $Z$ be the set of all chains in $X$, ordered by inclusion. If $C \in Z$, then every subset of $C$ is in $Z$, too. Moreover, the union of every chain in $Z$ is again an element of $Z$. 

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The Axiom of Choice
**Theorem. Zorn’s Lemma.** Let $X$ be a nonempty ordered set so that every chain in $X$ has an upper bound. Then $X$ has a maximal element.

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Theorem. Zorn’s Lemma. Let \( X \) be a nonempty ordered set so that every chain in \( X \) has an upper bound. Then \( X \) has a maximal element.

Proof. Let \( Z \) be the set of all chains in \( X \), ordered by inclusion. If \( C \in Z \), then every subset of \( C \) is in \( Z \), too. Moreover, the union of every chain in \( Z \) is again an element of \( Z \). Hence \( Z \) has a maximal element \( M \) with respect to inclusion. This set \( M \) has an upper bound \( m \) in \( X \), and \( M \cup \{m\} \) is a chain in \( X \), that is, \( M \cup \{m\} \in Z \). But \( M \) is maximal in \( Z \), so \( m \in M \). Now let \( x \in X \) satisfy \( x \geq m \). 

Bernd Schröder
Choice Functions
Zorn’s Lemma
Well-Ordering Theorem

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The Axiom of Choice
The Axiom of Choice
**Theorem. Zorn’s Lemma.** Let $X$ be a nonempty ordered set so that every chain in $X$ has an upper bound. Then $X$ has a maximal element.

**Proof.** Let $Z$ be the set of all chains in $X$, ordered by inclusion. If $C \in Z$, then every subset of $C$ is in $Z$, too. Moreover, the union of every chain in $Z$ is again an element of $Z$. Hence $Z$ has a maximal element $M$ with respect to inclusion. This set $M$ has an upper bound $m$ in $X$, and $M \cup \{m\}$ is a chain in $X$, that is, $M \cup \{m\} \in Z$. But $M$ is maximal in $Z$, so $m \in M$. Now let $x \in X$ satisfy $x \geq m$. Then $M \cup \{x\} \in Z$. 
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Theorem. Zorn’s Lemma. Let $X$ be a nonempty ordered set so that every chain in $X$ has an upper bound. Then $X$ has a maximal element.

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Theorem.

Let $A$ be an infinite set.

$\left( A \times \{0\} \right) \cup \left( A \times \{1\} \right) = A \times \{0, 1\}$ is equivalent to $A$.

Proof. Let $F$ be the set of all bijective functions $f: X \times \{0, 1\} \to X$, where $X \subseteq A$. $F \neq / 0$, because it contains all the bijective functions $f: X \times \{0, 1\} \to X$, where $X \subseteq A$ is countable. $F$ is ordered by set inclusion. Moreover for any chain $C \subseteq F$ we can form the union $u = \bigcup C$, and it will be a bijective function $u: X_u \times \{0, 1\} \to X_u$ for some subset $X_u \subseteq A$ (good exercise). Now $u$ is an upper bound for $C$ in $F$. Thus the hypotheses of Zorn's Lemma are satisfied. Let $h: X \times \{0, 1\} \to X$ be a maximal element of $F$. Suppose for a contradiction that $A \setminus X$ contains a countably infinite set $C$. Let $b: C \times \{0, 1\} \to C$ be a bijective function. Then $t = h \cup b$ is a bijective function between $(X \cup C) \times \{0, 1\}$ and $X \cup C$, that is, $t \in F$, contradiction.
**Theorem.** Let $A$ be an infinite set.
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**Theorem.** Let $A$ be an infinite set. Then $(A \times \{0\}) \cup (A \times \{1\}) = A \times \{0, 1\}$ is equivalent to $A$.

**Proof.** Let $\mathcal{F}$ be the set of all bijective functions $f : X \times \{0, 1\} \to X$, where $X \subseteq A$. 
**Theorem.** Let $A$ be an infinite set. Then

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**Proof.** Let $\mathcal{F}$ be the set of all bijective functions $f : X \times \{0, 1\} \to X$, where $X \subseteq A$. $\mathcal{F} \neq \emptyset$, because it contains all the bijective functions $f : X \times \{0, 1\} \to X$, where $X \subseteq A$ is countable.
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Theorem. Let $A$ be an infinite set. Then
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**Theorem.** Let $A$ be an infinite set. Then 
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**Proof.** Let \( \mathcal{F} \) be the set of all bijective functions \( f : X \times \{0, 1\} \rightarrow X \), where \( X \subseteq A \). \( \mathcal{F} \neq \emptyset \), because it contains all the bijective functions \( f : X \times \{0, 1\} \rightarrow X \), where \( X \subseteq A \) is countable. \( \mathcal{F} \) is ordered by set inclusion. Moreover for any chain \( \mathcal{C} \subseteq \mathcal{F} \) we can form the union \( u := \bigcup \mathcal{C} \), and it will be a bijective function \( u : X_u \times \{0, 1\} \rightarrow X_u \) for some subset \( X_u \subseteq A \) (good exercise). Now \( u \) is an upper bound for \( \mathcal{C} \) in \( \mathcal{F} \).
**Theorem.** Let $A$ be an infinite set. Then 
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Theorem. Let \( A \) be an infinite set. Then
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Proof. Let \( \mathcal{F} \) be the set of all bijective functions 
\[ f : X \times \{0, 1\} \to X, \text{ where } X \subseteq A. \] \( \mathcal{F} \neq \emptyset, \) because it contains all the bijective functions \( f : X \times \{0, 1\} \to X, \text{ where } X \subseteq A \) is countable. \( \mathcal{F} \) is ordered by set inclusion. Moreover for any chain \( \mathcal{C} \subseteq \mathcal{F} \) we can form the union \( u := \bigcup \mathcal{C} \), and it will be a bijective function \( u : X_u \times \{0, 1\} \to X_u \) for some subset \( X_u \subseteq A \) (good exercise). Now \( u \) is an upper bound for \( \mathcal{C} \) in \( \mathcal{F} \). Thus the hypotheses of Zorn’s Lemma are satisfied.
Let \( h : X \times \{0, 1\} \to X \) be a maximal element of \( \mathcal{F} \).
Theorem. Let $A$ be an infinite set. Then
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Let $h : X \times \{0, 1\} \rightarrow X$ be a maximal element of $\mathcal{F}$. Suppose for a contradiction that $A \setminus X$ contains a countably infinite set $C$. 
**Theorem.** Let $A$ be an infinite set. Then
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$$(A \times \{0\}) \cup (A \times \{1\}) = A \times \{0, 1\}$$ is equivalent to $A$.

**Proof.** Let $\mathcal{F}$ be the set of all bijective functions $f : X \times \{0, 1\} \to X$, where $X \subseteq A$. $\mathcal{F} \neq \emptyset$, because it contains all the bijective functions $f : X \times \{0, 1\} \to X$, where $X \subseteq A$ is countable. $\mathcal{F}$ is ordered by set inclusion. Moreover for any chain $C \subseteq \mathcal{F}$ we can form the union $u := \bigcup C$, and it will be a bijective function $u : X_u \times \{0, 1\} \to X_u$ for some subset $X_u \subseteq A$ (good exercise). Now $u$ is an upper bound for $C$ in $\mathcal{F}$. Thus the hypotheses of Zorn’s Lemma are satisfied.

Let $h : X \times \{0, 1\} \to X$ be a maximal element of $\mathcal{F}$. Suppose for a contradiction that $A \setminus X$ contains a countably infinite set $C$.

Let $b : C \times \{0, 1\} \to C$ be a bijective function. Then $t : h \cup b$ is a bijective function between $(X \cup C) \times \{0, 1\}$ and $X \cup C$. 
**Theorem.** Let $A$ be an infinite set. Then 
$$(A \times \{0\}) \cup (A \times \{1\}) = A \times \{0, 1\}$$ is equivalent to $A$.

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**Theorem.** Let $A$ be an infinite set. Then 
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Proof (cont.). Therefore $A \setminus X$ cannot be infinite. If $A \setminus X = \emptyset$, then the function $h$ is the desired bijection between $A \times \{0, 1\}$ and $A$. Finally consider the case that $A \setminus X \neq \emptyset$. By the above, $A \setminus X$ is finite. Let $C \subseteq X$ be a countably infinite subset of $X$. Let $R \subseteq C$ be an $|A \setminus X|$-element subset of $C$. Then $C \setminus R$ is still countably infinite. Let $p : h^{-1}[C] \rightarrow C \setminus R$ be a bijective function and let $q : (A \setminus X) \times \{0, 1\} \rightarrow A \setminus X \cup R$ be a bijective function. Then $t := h \setminus h|_{h^{-1}[C]} \cup p \cup q$ is the desired bijective function with domain $A \times \{0, 1\}$ and range $A$. \hfill \blacksquare
Definition.
Definition. Let $S$ be a set and let $\leq \subseteq S \times S$ be an order relation.
**Definition.** Let $S$ be a set and let $\leq \subseteq S \times S$ be an order relation. Then $\leq$ is called a **well-order** (relation) iff it is a total order.
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**Example.**
**Definition.** Let $S$ be a set and let $\leq \subseteq S \times S$ be an order relation. Then $\leq$ is called a **well-order** (relation) iff it is a total order and every nonempty subset of $S$ has a smallest element with respect to $\leq$.

**Example.** $\mathbb{N}$ is well-ordered.
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**Theorem.** Well-ordering Theorem.
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**Theorem.** **Well-ordering Theorem.** *Every set can be well-ordered.*
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**Example.** \( \mathbb{N} \) is well-ordered.

**Theorem.** **Well-ordering Theorem.** Every set can be well-ordered. That is, for every set $S$, there is a well-order relation $\leq \subseteq S \times S$. 
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The Axiom of Choice
Proof. Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$. 
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Proof. Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 \mid_{D_1 \times D_1} = \leq_1$. Then $\sqsubseteq$ is an order relation on $X$ (good exercise).
**Proof.** Let $X$ be the set of all well-order relations $\leq \subset D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subset D_1 \times D_1$ and $\leq_2 \subset D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 \upharpoonright D_1 \times D_1 = \leq_1$. Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$. 

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Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$.

$\leq$ is an order relation: Reflexivity is trivial.
**Proof.** Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 \mid_{D_1 \times D_1} = \leq_1$.

Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$.

$\leq$ is an order relation: Reflexivity is trivial. Let $D$ be the domain of the relation $\leq$. 
Proof. Let \( X \) be the set of all well-order relations \( \leq \subseteq D \times D \), where \( D \) is a subset of \( S \). Then \( X \neq \emptyset \). For any two well-order relations \( \leq_1 \subseteq D_1 \times D_1 \) and \( \leq_2 \subseteq D_2 \times D_2 \) in \( X \) define \( \leq_1 \subseteq \leq_2 \) iff \( D_1 \subseteq D_2 \), every \( d_2 \in D_2 \setminus D_1 \) is a strict \( \leq_2 \)-upper bound of \( D_1 \), and \( \leq_2 \mid_{D_1 \times D_1} = \leq_1 \). Then \( \subseteq \) is an order relation on \( X \) (good exercise). Let \( \mathcal{C} \subseteq X \) be a chain and let \( \leq := \bigcup \mathcal{C} \).

\( \leq \) is an order relation: Reflexivity is trivial. Let \( D \) be the domain of the relation \( \leq \). For antisymmetry, let \( x, y \in D \) be so that \( x \leq y \) and \( y \leq x \).
**Proof.** Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 \mid_{D_1 \times D_1} = \leq_1$.

Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$.

$\leq$ is an order relation: Reflexivity is trivial. Let $D$ be the domain of the relation $\leq$. For antisymmetry, let $x, y \in D$ be so that $x \leq y$ and $y \leq x$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $x, y \in D'$. 
**Proof.** Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 |_{D_1 \times D_1} = \leq_1$.

Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$.

$\leq$ is an order relation: Reflexivity is trivial. Let $D$ be the domain of the relation $\leq$. For antisymmetry, let $x, y \in D$ be so that $x \leq y$ and $y \leq x$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $x, y \in D'$. Hence $x \leq' y$ and $y \leq' x$, which implies $x = y$. 
**Proof.** Let $X$ be the set of all well-order relations $\leq \subseteq D \times D$, where $D$ is a subset of $S$. Then $X \neq \emptyset$. For any two well-order relations $\leq_1 \subseteq D_1 \times D_1$ and $\leq_2 \subseteq D_2 \times D_2$ in $X$ define $\leq_1 \sqsubseteq \leq_2$ iff $D_1 \subseteq D_2$, every $d_2 \in D_2 \setminus D_1$ is a strict $\leq_2$-upper bound of $D_1$, and $\leq_2 \upharpoonright D_1 \times D_1 = \leq_1$.

Then $\sqsubseteq$ is an order relation on $X$ (good exercise). Let $\mathcal{C} \subseteq X$ be a chain and let $\leq := \bigcup \mathcal{C}$.

$\leq$ is an order relation: Reflexivity is trivial. Let $D$ be the domain of the relation $\leq$. For antisymmetry, let $x, y \in D$ be so that $x \leq y$ and $y \leq x$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $x, y \in D'$. Hence $x \leq' y$ and $y \leq' x$, which implies $x = y$. Transitivity is proved similarly.
Proof (cont.).

Now let $\leq' \in C$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$.

Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in C$ with domain $D''$ so that $\leq' \sqsubseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$.

Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup C$, $\leq|_{D' \times D'} = \leq'$.

This does not establish that $\leq$ is an upper bound of $C$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$.

Then there is a $\leq' \in C$ with domain $D'$ so that $A \cap D' \neq \emptyset$.

Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$.

Because $\leq|_{D' \times D'} = \leq'$, $a$ is the $\leq$-smallest element of $A \cap D'$.

Because all elements of $D \setminus D'$ are $\leq$-strict upper bounds of $D'$, $a$ is the $\leq$-smallest element of $A$.

Therefore (simple exercise, maybe too simple) $\leq$ is a well-order. Hence it is a $\sqsubseteq$-upper bound of $C$. 

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Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. 
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Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq |_{D' \times D'} = \leq'$. 

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Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \sqsubseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}, \leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$. 
Proof (cont.). Now let $\leq' \in \mathbb{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathbb{C}$ with domain $D''$ so that $\leq' \sqsubseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathbb{C}$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathbb{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. 
Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $A \cap D' \neq \emptyset$. 
Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $A \cap D' \neq \emptyset$. Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$. 
Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \sqsubseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq|_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $A \cap D' \neq \emptyset$. Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$. Because $\leq|_{D' \times D'} = \leq'$, $a$ is the $\leq$-smallest element of $A \cap D'$. 
Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $A \cap D' \neq \emptyset$. Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$. Because $\leq |_{D' \times D'} = \leq'$, $a$ is the $\leq$-smallest element of $A \cap D'$. Because all elements of $D \setminus D'$ are $\leq$-strict upper bounds of $D'$, $a$ is the $\leq$-smallest element of $A$. 

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**Proof (cont.).** Now let $\leq' \in C$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in C$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup C$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $C$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in C$ with domain $D'$ so that $A \cap D' \neq \emptyset$. Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$. Because $\leq |_{D' \times D'} = \leq'$, $a$ is the $\leq$-smallest element of $A \cap D'$. Because all elements of $D \setminus D'$ are $\leq$-strict upper bounds of $D'$, $a$ is the $\leq$-smallest element of $A$. Therefore (simple exercise, maybe too simple) $\leq$ is a well-order.
Proof (cont.). Now let $\leq' \in \mathcal{C}$ and let $D'$ be the domain of $\leq'$. Clearly, $D' \subseteq D$. Let $d \in D \setminus D'$ and let $d' \in D'$. There is a $\leq'' \in \mathcal{C}$ with domain $D''$ so that $\leq' \subseteq \leq''$ and $d \in D'' \setminus D'$. But then $d \geq'' d'$, which means $d > d'$. Hence $d$ is a strict $\leq$-upper bound of $D'$. Finally, because $\leq = \bigcup \mathcal{C}$, $\leq |_{D' \times D'} = \leq'$. This does not establish that $\leq$ is an upper bound of $\mathcal{C}$, because we still do not know if $\leq \in X$.

For $\leq \in X$, let $A \subseteq D$ be a nonempty subset of $D$. Then there is a $\leq' \in \mathcal{C}$ with domain $D'$ so that $A \cap D' \neq \emptyset$. Because $\leq'$ is a well-order, $A \cap D'$ has a $\leq'$-smallest element $a$. Because $\leq |_{D' \times D'} = \leq'$, $a$ is the $\leq$-smallest element of $A \cap D'$. Because all elements of $D \setminus D'$ are $\leq$-strict upper bounds of $D'$, $a$ is the $\leq$-smallest element of $A$. Therefore (simple exercise, maybe too simple) $\leq$ is a well-order. Hence it is a $\subseteq$-upper bound of $\mathcal{C}$.
Proof (concl.).
Proof (concl.). By Zorn’s Lemma, $X$ has a $\subseteq$-maximal element $\leq$. 
Proof (concl.). By Zorn’s Lemma, $X$ has a $\sqsubseteq$-maximal element $\leq$. Then $\leq$ is a well-order with domain $D$. 

\[ \text{Proof (concl.). By Zorn’s Lemma, } X \text{ has a } \sqsubseteq-\text{maximal element } \leq. \text{ Then } \leq \text{ is a well-order with domain } D. \]
**Proof (concl.).** By Zorn’s Lemma, $X$ has a $\sqsubseteq$-maximal element $\leq$. Then $\leq$ is a well-order with domain $D$. Suppose for a contradiction that $D \neq S$ and let $s \in D \setminus S$. 
Proof (concl.). By Zorn’s Lemma, $X$ has a $\sqsubseteq$-maximal element $\leq$. Then $\leq$ is a well-order with domain $D$. Suppose for a contradiction that $D \neq S$ and let $s \in D \setminus S$. Define $\leq'$ to be an order relation on $D \cup \{s\}$ so that $\leq' |_{D \times D} = \leq$ and so that $s$ is a strict $\leq'$-upper bound of $D$. Then $\leq' \in X$ is a strict $\sqsubseteq$-upper bound of $\leq$, contradicting the maximality of $\leq$. Hence $\leq$ must be a well-order for $S$. 
Proof (concl.). By Zorn’s Lemma, $X$ has a $\sqsubseteq$-maximal element $\leq$. Then $\leq$ is a well-order with domain $D$. Suppose for a contradiction that $D \neq S$ and let $s \in D \setminus S$. Define $\leq'$ to be an order relation on $D \cup \{s\}$ so that $\leq'|_{D \times D} = \leq$ and so that $s$ is a strict $\leq'$-upper bound of $D$. Then $\leq' \in X$ is a strict $\sqsubseteq$-upper bound of $\leq$, contradicting the maximality of $\leq$. 
Proof (concl.). By Zorn’s Lemma, $X$ has a $\sqsubseteq$-maximal element $\leq$. Then $\leq$ is a well-order with domain $D$. Suppose for a contradiction that $D \neq S$ and let $s \in D \setminus S$. Define $\leq'$ to be an order relation on $D \cup \{s\}$ so that $\leq' |_{D \times D} = \leq$ and so that $s$ is a strict $\leq'$-upper bound of $D$. Then $\leq' \in X$ is a strict $\sqsubseteq$-upper bound of $\leq$, contradicting the maximality of $\leq$. Hence $\leq$ must be a well-order for $S$. 

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**Proof (concl.).** By Zorn’s Lemma, \( X \) has a \( \sqsubseteq \)-maximal element \( \leq \). Then \( \leq \) is a well-order with domain \( D \). Suppose for a contradiction that \( D \neq S \) and let \( s \in D \setminus S \). Define \( \leq' \) to be an order relation on \( D \cup \{s\} \) so that \( \leq' \mid_{D \times D} = \leq \) and so that \( s \) is a strict \( \leq' \)-upper bound of \( D \). Then \( \leq' \in X \) is a strict \( \sqsubseteq \)-upper bound of \( \leq \), contradicting the maximality of \( \leq \). Hence \( \leq \) must be a well-order for \( S \).