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GRADUATE SCHOOL
PHYSICS

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MESHLESS NUMERICAL METHODS

Project B

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1 Project B

Consider the following convection-diffusion-reaction equation

\[ k \Delta u(x, y) + (x^2 + y^2)u + y \cos(y) \frac{\partial u}{\partial x} + \sinh(x) \frac{\partial u}{\partial y} = \]

\[ (k (1 - \pi^2) + (x^2 + y^2)) [\sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y)] + \]

\[ + \pi y \cos(y) [\cos(\pi x) \cosh(y) + \sin(\pi x) \sinh(y)] + \]

\[ + \sinh(x) [\sin(\pi x) \sinh(y) - \cos(\pi x) \cosh(y)] \quad \text{in} \quad \Omega \]

(1)

\[ u(x, y) = \sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y) \quad \text{on} \quad \partial \Omega \]

(2)

where \( \Omega \cup \partial \Omega = (x, y) : 0.0 \leq x \leq 1.0, 0.0 \leq y \leq 1.0 \). The exact solution of above equations is given by

\[ u(x, y) = \sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y). \]

(3)

Using the method of particular solutions with MQ as a basis function for interpolating the right hand side of Poisson’s equation, solve the above problem using 361 random points inside the domain and 80 boundary points. Test the result using various shape parameters \( c \) and then find the maximum error at another 225 random points inside the domain and 60 on the boundary.

2 Method

The method of approximate particular solution [1] tries to find solution of elliptic partial differential equation in the form of

\[ \hat{u}(x, y) = \sum_{i=1}^{n} a_i \Phi(r_i), \]

(4)

where

\[ \Delta \Phi = \phi. \]

(5)

\( \phi \) are the radial basis functions and \( \Phi \) their corresponding particular solutions, \( n = n_{in} + n_b \) is the sum of all \( n_{in} \) points in the domain \( \Omega \) and all \( n_b \) points on the boundary \( \partial \Omega \). It then holds true that

\[ \Delta u(x, y) \simeq \Delta \hat{u}(x, y) = \sum_{i=1}^{n} a_i \Delta \Phi(r_i) = \sum_{i=1}^{n} a_i \phi(r_i) \]

(6)
We can then write our problem with two sets of equations. \( n_{in} \) equations for chosen domain

\[
\sum_{i=1}^{n} a_i \left[ k\phi(r_{ij}) + (x_j^2 + y_j^2)\Phi(r_{ij}) + y_j \cos(y_j) \frac{\partial \Phi(r_{ij})}{\partial x} + \sinh(x_j) \frac{\partial \Phi(r_{ij})}{\partial y} \right] = \\
(k \left( 1 - \pi^2 \right) + (x_j^2 + y_j^2)) \left[ \sin(\pi x_j) \cosh(y_j) - \cos(\pi x_j) \sinh(y_j) \right] + \\
+ \pi y_j \cos(y_j) \left[ \cos(\pi x_j) \cosh(y_j) + \sin(\pi x_j) \sinh(y_j) \right] + \\
+ \sinh(x_j) \left[ \sin(\pi x_j) \sinh(y_j) - \cos(\pi x_j) \cosh(y_j) \right], \quad 1 \leq j \leq n_{in}
\]

(7)

and \( n_b \) for boundary

\[
\sum_{i=1}^{n} a_i \Phi(r_{ij}) = \sin(\pi x_j) \cosh(y_j) - \cos(\pi x_j) \sinh(y_j), \quad n_{in} + 1 \leq j \leq n, \quad (8)
\]

where \( r_{ij} = \|(x_j, y_j) - (x_i, y_i)\| \).

If we rewrite Equations (7) and (8) in the form of

\[
\sum_{i=1}^{n} a_i \left[ M1(x_j, y_j, r_{ij}) \right] = b1(x_j, y_j), \quad 1 \leq j \leq n_{in}
\]

(9)

and

\[
\sum_{i=1}^{n} a_i \Phi(r_{ij}) = b2(x_j, y_j), \quad n_{in} + 1 \leq j \leq n, \quad (10)
\]

we can then write our problem in the form of matrices like

\[
\begin{pmatrix}
M1(x_1, y_1, r_{11}) & M1(x_1, y_1, r_{21}) & \cdots & M1(x_1, y_1, r_{n_{in}}) \\
M1(x_2, y_2, r_{12}) & M1(x_2, y_2, r_{22}) & \cdots & M1(x_2, y_2, r_{n_{in}}) \\
\vdots & \vdots & \ddots & \vdots \\
M1(x_{n_{in}}, y_{n_{in}}, r_{1n_{in}}) & M1(x_{n_{in}}, y_{n_{in}}, r_{2n_{in}}) & \cdots & M1(x_{n_{in}}, y_{n_{in}}, r_{n_{in}_{in}}) \\
\Phi(r_{1n_{in}+1}) & \Phi(r_{2n_{in}+1}) & \cdots & \Phi(r_{n_{in}_{in}+1}) \\
\Phi(r_{1n_{in}+2}) & \Phi(r_{2n_{in}+2}) & \cdots & \Phi(r_{n_{in}_{in}+2}) \\
\vdots & \vdots & \ddots & \vdots \\
\Phi(r_{1n}) & \Phi(r_{2n}) & \cdots & \Phi(r_{n_{n}})
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_{n_{in}} \\
a_{n_{in}+1} \\
a_{n_{in}+2} \\
\vdots \\
a_n
\end{pmatrix}
\begin{pmatrix}
b1(x_1, y_1) \\
b1(x_2, y_2) \\
\vdots \\
b1(x_{n_{in}}, y_{n_{in}}) \\
b2(x_{n_{in}+1}, y_{n_{in}+1}) \\
b2(x_{n_{in}+2}, y_{n_{in}+2}) \\
\vdots \\
b2(x_n, y_n)
\end{pmatrix}
\]

For RBFs we use multiquadric functions (MQ)

\[
\phi(r) = \sqrt{r^2 + c^2}
\]

(11)

and their corresponding particular solutions

\[
\Phi(r) = \frac{1}{9}(4c^2 + r^2)\sqrt{r^2 + c^2} - \frac{c^3}{3} \ln \left( c^2 + c\sqrt{r^2 + c^2} \right).
\]

(12)
The derivatives of particular solutions with respect to \( x \) and \( y \) look like

\[
\frac{\partial \Phi(r_{ij})}{\partial x} = (x_i - x_j) \left[ \frac{c \sqrt{r_{ij}^2 + c^2 + r_{ij}^2 + 2c^2}}{3 \left( c + \sqrt{r_{ij}^2 + c^2} \right)} \right]
\]

(13)

and

\[
\frac{\partial \Phi(r_{ij})}{\partial y} = (y_i - y_j) \left[ \frac{c \sqrt{r_{ij}^2 + c^2 + r_{ij}^2 + 2c^2}}{3 \left( c + \sqrt{r_{ij}^2 + c^2} \right)} \right]
\]

(14)

Now we have to solve this system of equations \( A \vec{a} = \vec{b} \) and then with coefficients \( a_i \) write the solution of our problem in the form of Equation (4).

## 3 Results

To solve linear system of equations I used the Singular value decomposition method where I equated to zero all the singular values that are smaller than \((\text{maximal singular value}) \cdot 10^{(-16+16/5)}\). In this way I got rid of all the singular values that are affected by the precision of the double precision arithmetic, but still tried to use as many values as possible that still did not corrupt the results.

I used \( n_{in} = 361 \) random points in computational domain \( \Omega \), \( n_b = 80 \) points on it’s boundary \( \partial \Omega \) (Figure 1) and another \( n_t = 225 \) test points inside the domain and \( n_r = 60 \) test points on it’s boundary.

By tracking the root-mean-square error (RMSE)

\[
RMSE = \sqrt{\frac{1}{n_t} \sum_{j=1}^{n_t} (\hat{u}_j - u_j)^2}
\]

(15)

and residual of Dirichlet boundary conditions

\[
ResidualD = \frac{1}{n_r} \sum_{j=1}^{n_r} |\hat{u}_j - g_j|
\]

(16)

where \( g_j \) is is the given boundary condition at the \( j^{th} \) point (Equation 2), with respect to shape parameter \( c \) for additional \( n_t \) and \( n_r \) test points (Figure 2) I could find the optimal shape parameter \( c \) for different sets of numbers (Table 1). I chose optimal shape parameters with respect to minimal RMSE.
Figure 1: \( n_{in} \) random points in computational domain and \( n_b \) points on its boundary.

Figure 2: RMSE and ResidualD with respect to shape parameter \( c \).
Table 1: Optimal shape parameters $c$ at different sets of numbers.

<table>
<thead>
<tr>
<th>$n_1, n$</th>
<th>$n_b$</th>
<th>optimal $c$</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>361</td>
<td>80</td>
<td>0.49</td>
<td>$3.8E-6$</td>
</tr>
<tr>
<td>289</td>
<td>68</td>
<td>0.59</td>
<td>$4.2E-6$</td>
</tr>
<tr>
<td>196</td>
<td>40</td>
<td>0.57</td>
<td>$9.3E-6$</td>
</tr>
</tbody>
</table>

The solution of our problem (Equations 1 and 2) at optimal shape parameter $c = 0.49$ is presented in Figure 3.

Figure 3: The profile of solution of our problem.

Literatura