These notes correspond to Section 8.6 in the text.

**Differentiation and Integration of Power Series**

We have previously learned how to compute power series representations of certain functions, by relating them to geometric series. We can obtain power series representation for a wider variety of functions by exploiting the fact that a convergent power series can be differentiated, or integrated, term-by-term to obtain a new power series that has the same radius of convergence as the original power series. The new power series is a representation of the derivative, or antiderivative, of the function that is represented by the original power series.

This is particularly useful when we have a function \( f(x) \) for which we do not know how to obtain a power series representation directly. If its derivative \( f'(x) \), or its antiderivative \( \int f(x) \, dx \), is a function for which a power series representation can easily be computed, such as the examples from the previous lecture, then we can integrate, or differentiate, this power series term-by-term to obtain a power series for \( f(x) \).

**Example** The function

\[
 f(x) = \frac{4}{(2 - x)^2}
\]

is the derivative of the function

\[
 g(x) = \frac{2x}{2 - x},
\]

which, from the previous lecture, has the power series representation

\[
 \frac{2x}{2 - x} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} x^n.
\]

This series converges when \(-2 < x < 2\). To obtain a power series representation of \( f(x) \), we differentiate this series term-by-term to obtain

\[
 \frac{4}{(2 - x)^2} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} n x^{n-1} = \sum_{n=0}^{\infty} \frac{(n + 1)}{2^n} x^n,
\]

which also converges when \(-2 < x < 2\). \(\Box\)
**Example** The function

\[ f(x) = \frac{1}{2} \tan^{-1} \frac{x - 2}{2} \]

has the derivative

\[ f'(x) = \frac{1}{(x - 2)^2 + 4}. \]

From the previous lecture, this function has the power series

\[ \frac{1}{(x - 2)^2 + 4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} (x - 2)^{2n}, \]

whose interval of convergence is \(0 < x < 4\). Integrating this series term-by-term yields

\[ \frac{1}{2} \tan^{-1} \frac{x - 2}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \int (x - 2)^{2n} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \frac{(x - 2)^{2n+1}}{2n+1} + C. \]

To determine the value of \(C\), we substitute \(x = 2\) into the above equation. This causes all terms in the series to vanish. We also have \(f(2) = 0\), which yields \(C = 0\). □

**Example** Consider the definite integral

\[ \int_{0}^{1} \frac{1}{1 + x^4} \, dx. \]

Attempting to evaluate this integral using partial fraction decomposition is not possible without introducing complex numbers. Instead, we express the integrand as a (geometric) power series:

\[ \frac{1}{1 + x^4} = \frac{1}{1 - (-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}. \]

This power series has an interval of convergence of \(-1 < x < 1\), which contains the interval of integration \((0, 1)\). Integrating the power series term-by-term from 0 to 1 yields

\[ \int_{0}^{1} \frac{1}{1 + x^4} \, dx = \int_{0}^{1} \sum_{n=0}^{\infty} (-1)^n x^{4n} \, dx = \sum_{n=0}^{\infty} (-1)^n \int_{0}^{1} x^{4n} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} \bigg|_{0}^{1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1}. \]

This is an alternating series, which, by the Alternating Series Test, converges since, for all \(n \geq 0\),

\[ \frac{1}{4n + 1} \geq 0, \quad \lim_{n \to \infty} \frac{1}{4n + 1} = 0, \quad \text{and} \quad \frac{1}{4(n + 1) + 1} < \frac{1}{4n + 1}. \]

Using the Alternating Series Estimation Theorem, we can evaluate this integral numerically, to any degree of accuracy we wish, by choosing \(n\) large enough so that \(1/(4n + 1)\) is sufficiently small. □
Summary

- A power series representation of a function $f(x)$ can be differentiated term-by-term to obtain a power series representation of its derivative $f'(x)$. The interval of convergence of the differentiated series is the same as that of the original series.

- A power series representation of a function $f(x)$ can be anti-differentiated term-by-term to obtain a power series representation of its anti-derivative $\int f(x) \, dx$. The value of the constant of integration, $C$, can be determined by substituting the center of the power series for $x$. The interval of convergence of the anti-differentiated series is the same as that of the original series.

- A power series representation of a function $f(x)$ can be integrated term-by-term from $a$ to $b$ to obtain a series representation of the definite integral $\int_a^b f(x) \, dx$, provided that the interval $(a, b)$ lies within the interval of convergence of the power series that represents $f(x)$. 