These notes correspond to Section 10.5 in the text.

Equations of Planes

Previously, we learned how to describe lines using various types of equations. Now, we will do the same with planes. Suppose that we are given three points \( r_0, r_1 \) and \( r_2 \) that are not co-linear. Then, these points define a plane, and the vectors \( v_1 = r_1 - r_0 \) and \( v_2 = r_2 - r_0 \) are vectors contained within the plane, that are also not parallel to one another.

A plane consists of all vectors that are orthogonal to a given direction \( n \), which is said to be normal to the plane, and passes through a given point \( r_0 \). The normal vector \( n \) can be obtained by computing

\[
    n = v_1 \times v_2.
\]

Let \( r \) be any point in the plane. Then the vector \( u = r - r_0 \) is orthogonal to \( n \). That is,

\[
    n \cdot u = n \cdot (r - r_0) = 0.
\]

This equation is called the vector equation of the plane.

If we write

\[
    n = (a, b, c), \quad r = (x, y, z), \quad r_0 = (x_0, y_0, z_0),
\]

then the vector equation can be rewritten as

\[
    ax + by + cz + d = 0,
\]

where \( d = -n \cdot r_0 = -(ax_0 + by_0 + cz_0) \). This is a linear equation in the unknowns \( x, y, \) and \( z \).

**Solution** Consider the plane containing the points \( P_0 = (1, 4, 1), P_1 = (5, 1, -1) \) and \( P_2 = (4, 4, 4) \), which we identify with the position vectors

\[
    r_0 = (1, 4, 1), \quad r_1 = (5, 1, -1), \quad r_2 = (4, 4, 4).
\]

We wish to find a linear equation that describes this plane. First, we need to compute a vector \( n \) that is normal to the plane, which can be obtained by computing the cross product of two vectors \( v_1 \) and \( v_2 \) that are contained within the plane. We have

\[
    n = v_1 \times v_2 = (r_1 - r_0) \times (r_2 - r_0)
\]
\[
\langle 4, -3, -2 \rangle \times \langle 3, 0, 3 \rangle \\
= \langle -3(3) - (-2)0, -2(3) - 4(3), 4(0) - (-3)(3) \rangle \\
= \langle -9, -18, 9 \rangle.
\]

It follows that the vector equation of the plane is

\[
\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \langle -9, -18, 9 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 4, 1 \rangle) = 0,
\]

which can also be written as

\[-9x - 18y + 9z + 72 = 0,
\]

since

\[
\mathbf{n} \cdot \mathbf{r}_0 = -72.
\]

\[\square\]

**Intersecting Planes**

The *angle between planes* is defined to be the angle between their normal vectors. If this angle is either 0 or \(\pi\), then the normal vectors are parallel, and we say that the planes are parallel. Otherwise, the planes intersect, and this intersection is a line.

To determine the line formed by this intersection, we need to solve the system of equations consisting of the equations of the two planes. Because this system of equations has three unknowns, but there are only two equations, there will be infinitely many points that satisfy the system, and the set of all such solutions constitutes a line.

Let the equations of two planes be given by

\[a_1 x + b_1 y + c_1 z + d_1 = 0, \quad a_2 x + b_2 y + c_2 z + d_2 = 0,\]

and let the corresponding normal vectors be

\[\mathbf{n}_1 = \langle a_1, b_1, c_1 \rangle, \quad \mathbf{n}_2 = \langle a_2, b_2, c_2 \rangle.\]

To solve this system of equations, we first check whether the equations are independent if we were to set \(z = 0\). That is, we must check whether

\[a_1 b_2 - b_1 a_2 = 0,\]

or, equivalently, whether \(a_1\) and \(b_1\) are proportional to \(a_2\) and \(b_2\). If they are *not* proportional, then we can set \(z = 0\) to obtain the system of equations

\[a_1 x + b_1 y = -d_1, \quad a_2 x + b_2 y = -d_2,\]
which is now guaranteed to have a unique solution. This gives us a point on the line that is common to both planes. If \( a_1b_2 - b_1a_2 = 0 \), then we cannot necessarily substitute \( z = 0 \), for the resulting system of equations might be inconsistent. Instead, we can set \( y = 0 \), in which case the resulting system of equations, for the unknowns \( x \) and \( z \), will have a unique solution.

To determine the direction of the line of intersection, we note that any vector in a plane is orthogonal to its normal vector. Because this line belongs to both planes, a vector in the direction of the line is orthogonal to both normal vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \). It follows that a vector \( \mathbf{v} \) in the direction of the line of intersection can be found by computing

\[
\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2.
\]

This direction, and the previously computed point on the line, can be used to obtain the parametric or symmetric equations of the line.

**Example** Consider two planes defined by the equations

\[
x + 3y - 2z + 10 = 0, \quad 2x - 4y + 3z - 5 = 0.
\]

These planes are not parallel, because their normal vectors \( \mathbf{n}_1 = (1, 3, -2) \) and \( \mathbf{n}_2 = (2, -4, 3) \) are not parallel. Their intersection is a line that is parallel to the vector

\[
\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (1, 3, -2) \times (2, -4, 3) = (3(-4) - (-2)(-4), (-2)(2) - 1(3), 1(-4) - 3(2)) = (1, -7, -10).
\]

To write down the equation of the line of intersection, we need to compute the coordinates of a point on the line. Substituting \( z = 0 \) into the equations of the plane yields the system

\[
\begin{align*}
x + 3y &= -10, \\
2x - 4y &= 5.
\end{align*}
\]

This system has a unique solution, because the coefficients of the equations are not proportional. Subtracting twice the first equation from the second yields the simpler equation \(-10y = 25\), so \( y = -5/2 \). Substituting this value into the first equation yields \( x = -10 - 3(-5/2) = -5/2 \). We conclude that the line can be described using the parametric equations

\[
x = -5/2 + t, \quad y = -5/2 - 7t, \quad z = -10t.
\]

We can also describe the line using the symmetric equations

\[
\frac{x + 5/2}{1} = \frac{y + 5/2}{-7} = \frac{z}{-10}.
\]
Distance from a Point to a Plane

Let \( \mathbf{p}_1 = (x_1, y_1, z_1) \) be a position vector corresponding to a point \( P = (x_1, y_1, z_1) \). Let \( (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \) be the equation of a plane, where \( \mathbf{r}_0 = (x_0, y_0, z_0) \) is the position vector for a point \( R_0 = (x_0, y_0, z_0) \) in the plane, and \( \mathbf{n} \) is the plane’s normal vector. We consider the problem of computing the distance \( D \) between the point \( P_1 \) and this plane.

Intuitively, it makes sense to define this distance as the distance from \( P_1 \) to some point \( P_2 \) contained within the plane. However, we need to determine what a suitable point \( P_2 \) would be. We choose \( P_2 \) to be the best approximation of \( P_1 \) by a point in the plane, just as the vector projection of a vector \( \mathbf{v} \) onto a vector \( \mathbf{u} \) was previously defined to be the best approximation of \( \mathbf{v} \) by a vector that is parallel to \( \mathbf{u} \).

The key characteristic of the best approximation of the point \( P_1 \) by a point \( P_2 \) in the plane is that the error in this approximation, that is, the vector between \( P_1 \) and \( P_2 \), should be orthogonal to the plane. That is, this vector should be parallel to \( \mathbf{n} \), the normal to the plane. To determine the length of this vector, we form a triangle with the points \( \mathbf{p}_1, \mathbf{p}_2 \) and \( \mathbf{r}_0 \), where \( \mathbf{p}_2 \) is the position vector for the point \( P_2 \).

Because \( \mathbf{p}_1 - \mathbf{p}_2 \) is parallel to \( \mathbf{n} \), which is orthogonal to \( \mathbf{p}_2 - \mathbf{r}_0 \), this triangle is a right triangle, with the hypotenuse defined by \( \mathbf{p}_1 - \mathbf{r}_0 \). Therefore, we can use right triangle trigonometry to determine that the distance \( D \) is given by

\[
D = |\mathbf{v}_1| \cos \theta
\]

where \( \mathbf{v}_1 = \mathbf{p}_1 - \mathbf{r}_0 \) and \( \theta \) is the angle between \( \mathbf{v}_1 \) and \( \mathbf{n} \), with \( \mathbf{n} \) chosen so that \( 0 \leq \theta < \pi/2 \). It follows that

\[
D = \frac{|\mathbf{v}_1 \cdot \mathbf{n}|}{|\mathbf{n}|}.
\]

It is interesting to note that from this formula, we can see that \( D \) is also the absolute value of the scalar projection of \( \mathbf{v}_1 \) onto \( \mathbf{n} \), or, equivalently, the magnitude of the vector projection of \( \mathbf{v}_1 \) onto \( \mathbf{n} \).

If \( \mathbf{n} = (a, b, c) \), and we write the equation of the plane in the form \( ax + by + cz + d = 0 \), then we can express this distance as

\[
D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}},
\]

because the equations of the plane, \((\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0\) and \(ax + by + cz + d = 0\), are related by \( d = -\mathbf{r}_0 \cdot \mathbf{n} = -(ax_0 + by_0 + cz_0)\).

**Example** We wish to compute the distance \( D \) between the point \( P_1 = (4, 5, 6) \) and the plane described by the linear equation

\[
2x + 3y - 4z + 15 = 0.
\]
The normal vector for this plane is \( \mathbf{n} = \langle a, b, c \rangle = (2, 3, -4) \). It follows that the distance \( D \) is given by

\[
D = \frac{|2(4) + 3(5) - 4(6) + 15|}{\sqrt{2^2 + 3^2 + (-4)^2}} = \frac{14}{\sqrt{29}} \approx 2.6.
\]

\( \square \)

The formula for the distance between a point and a plane can be used to compute the distance between two parallel planes. The idea is to identify one point in the first plane, and then compute the distance between this point and the second plane. Because the planes are parallel, this distance will be the same, regardless of which point from the first plane is chosen.
Summary

- The vector equation of a plane is \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \), where \( \mathbf{n} \) is a vector that is normal to the plane, \( \mathbf{r} \) is any position vector in the plane, and \( \mathbf{r}_0 \) is a given position vector in the plane. The normal vector \( \mathbf{n} \) can be obtained by computing the cross product of any two non-parallel vectors in the plane.

- Two planes are parallel if and only if their normal vectors are parallel.

- If two planes are not parallel, their intersection is a line. The direction of the line is a vector that is orthogonal to the planes’ normal vectors. A point on the line can be found by finding a solution of the system of equations consisting of the equations of the planes, which can be accomplished by setting one of the coordinates equal to zero.

- The distance between a point \( \mathbf{p} \) and a plane \( \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \) is the absolute value of the dot product of the unit (normalized) normal vector \( \mathbf{n}/|\mathbf{n}| \) and the vector between \( \mathbf{p} \) and \( \mathbf{r}_0 \).