Arc Length of Parametrically Defined Curves

In the last lecture we learned how to compute the arc length of a curve described by an equation of the form $y = f(x)$, where $a \leq x \leq b$. The arc length $L$ of such a curve is given by the definite integral

$$L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.$$

Now, suppose that this curve can also be defined by parametric equations

$$x = g(t), \quad y = h(t), \quad (1)$$

where $c \leq t \leq d$. It follows that

$$y = h(t) = f(g(t)),$$

and therefore, by the Chain Rule,

$$\frac{dy}{dt} = h'(t) = f'(g(t))g'(t) = f'(g(t)) \frac{dx}{dt} = f'(x) \frac{dx}{dt}.$$

In the integral defining the arc length of the curve, we make the substitution $x = g(t)$ and obtain

$$L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx$$

$$= \int_{c}^{d} \sqrt{1 + [f'(g(t))]^2} g'(t) \, dt$$

$$= \int_{c}^{d} \sqrt{[g'(t)]^2 + [f'(g(t))g'(t)]^2} \, dt$$

$$= \int_{c}^{d} \sqrt{[g'(t)]^2 + [h'(t)]^2} \, dt$$

$$= \int_{c}^{d} \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \, dt.$$
It turns out that this formula for the arc length applies to any curve that is defined by parametric equations of the form (1), as long as $x$ and $y$ are differentiable functions of the parameter $t$. To derive the formula in the general case, one can proceed as in the case of a curve defined by an equation of the form $y = f(x)$, and define the arc length as the limit as $n \to \infty$ of the sum of the lengths of $n$ line segments whose endpoints lie on the curve.

**Example** Compute the length of the curve

$$x = 2\cos^2 \theta, \quad y = 2\cos \theta \sin \theta,$$

where $0 \leq \theta \leq \pi$.

**Solution** This curve is plotted in Figure 1; it is a circle of radius 1 centered at the point $(1,0)$. It follows that its length, which we will denote by $L$, is the circumference of the circle, which is $2\pi$. Using the arc length formula, we can obtain the same result as follows:

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$= \int_0^\pi \sqrt{(-4\cos \theta \sin \theta)^2 + (2\cos^2 \theta - 2\sin^2 \theta)^2} \, d\theta$$

$$= 2 \int_0^\pi \sqrt{(2\cos \theta \sin \theta)^2 + (\cos^2 \theta - \sin^2 \theta)^2} \, d\theta$$

$$= 2 \int_0^\pi \sqrt{4\cos^2 \theta \sin^2 \theta + \cos^4 \theta - 2\cos^2 \theta \sin^2 \theta + \sin^4 \theta} \, d\theta$$

$$= 2 \int_0^\pi \sqrt{\cos^4 \theta + 2\cos^2 \theta \sin^2 \theta + \sin^4 \theta} \, d\theta$$

$$= 2 \int_0^\pi \sqrt{1} \, d\theta$$

$$= 2 \int_0^\pi d\theta$$

$$= 2\pi.$$

*Note:* Double-angle and half-angle formulas could have been used in this example, but little would have been gained except during the differentiation stage, so I chose not to use them. □

**Example** Compute the length of the curve

$$x = t \sin t, \quad y = t \cos t$$

from $t = 0$ to $t = 2\pi$.  

2
Figure 1: Curve defined by $x = \cos^2 \theta$, $y = 2 \cos \theta \sin \theta$

**Solution** The length $L$ is given by the integral

$$L = \int_{0}^{2\pi} \sqrt{\sin t + t \cos t}^2 + \sqrt{(\cos t - t \sin t)^2} \, dt$$

$$= \int_{0}^{2\pi} \sqrt{\sin^2 t + 2 \sin t \cos t + t^2 \cos^2 t + \cos^2 t - 2 \sin t \cos t + t^2 \sin^2 t} \, dt$$

$$= \int_{0}^{2\pi} \sqrt{(\sin^2 t + \cos^2 t) + t^2 (\cos^2 t + \sin^2 t)} \, dt$$

$$= \int_{0}^{2\pi} \sqrt{1 + t^2} \, dt$$

$$= \int_{0}^{\tan^{-1}(2\pi)} \sqrt{1 + \tan^2 \theta \sec^2 \theta} \, d\theta$$

$$= \int_{0}^{\tan^{-1}(2\pi)} \sec^3 \theta \, d\theta$$

3
\[
\begin{align*}
&= \frac{1}{2} \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]^{\tan^{-1}(2\pi)}_0 \\
&= \frac{1}{2} \left[ \sqrt{1 + \tan^2 \theta} \tan \theta + \ln |\sqrt{1 + \tan^2 \theta} + \tan \theta| \right]^{\tan^{-1}(2\pi)}_0 \\
&= \frac{1}{2} \left[ \sqrt{1 + (2\pi)^2(2\pi)} + \ln |\sqrt{1 + (2\pi)^2 + 2\pi}| \right] \\
&\approx 21.2563.
\end{align*}
\]

The integral of $\sec^3 \theta$ is obtained using integration by parts, with $u = \sec \theta$ and $dv = \sec^2 \theta \, d\theta$. The curve is displayed in Figure 2. □

![Figure 2: Curve defined by $x = t \sin t$, $y = t \cos t$](image)
Summary

- If a curve is defined by parametric equations \( x = g(t), \ y = h(t) \) for \( c \leq t \leq d \), the arc length of the curve is the integral of \( \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{[g'(t)]^2 + [h'(t)]^2} \) from \( c \) to \( d \).