These notes correspond to Section 8.4 in the text.

**Other Convergence Tests**

In this lecture, we develop additional tests that, for many series, will enable us to quickly determine whether a given series converges or diverges. Although these new tests, like the Integral and Comparison Tests, can only tell us whether a series converges, as opposed to helping us compute its limit, they do offer us one advantage that the previous tests do not: they are applicable to series whose terms are not necessarily positive.

**The Alternating Series Test**

An *alternating series* is a series whose terms alternate signs, so that two consecutive terms always have opposite signs.

**Example** The series

\[
\sum_{n=0}^{\infty} \left( \frac{-1}{2} \right)^n,
\]

in addition to being a geometric series with \(a = 1\) and \(r = -\frac{1}{2}\), is an alternating series whose first few terms are

\[
a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{4}, \quad a_3 = -\frac{1}{8}.
\]

\[\square\]

Any alternating series has terms of the form \(a_n = (-1)^n b_n\), where \(b_n = |a_n| > 0\). In the preceding example, \(b_n = 1/2^n\).

Suppose that we have an alternating series with terms \(\{(-1)^n b_n\}_{n=0}^{\infty}\), for which \(b_n > 0\) for \(n \geq 0\), such that the terms are non-increasing in magnitude: \(b_{n+1} \leq b_n\) for \(n \geq 0\). While one would normally try to establish convergence or divergence by examining the sequence of partial sums, because of the alternating signs of the terms, we will instead examine *alternating* partial sums.

Specifically, consider the sequence of even-numbered partial sums:

\[
s_{2n} = b_0 - b_1 + b_2 + \cdots + b_{2n}
\]

\[
= (b_0 - b_1) + (b_2 - b_3) + \cdots + b_{2n}.
\]
Because \( b_n \geq b_{n+1} \), each quantity in parentheses is a non-negative number, which means \( s_{2n} \geq (b_0 - b_1) \) for \( n \geq 0 \). That is, the sequence of even-numbered partial sums is bounded below. A similar grouping of terms can be used to show that \( s_{2n} \leq b_0 \) for \( n \geq 0 \), so this sequence is also bounded above. In other words, it is bounded.

On the other hand, by the definition of a partial sum, we have

\[
s_{2n} = s_{2n-2} - b_{2n-1} + b_{2n} = s_{2n-2} - (b_{2n-1} - b_{2n}) \leq s_{2n-2},
\]

which shows that this sequence of partial sums is also non-increasing. That is, this sequence is monotonic. It follows from the Monotonic Sequence Theorem that the sequence is convergent. A similar procedure can be used to show that not only does the sequence of odd-numbered partial sums converge, but it converges to the same limit as the even-numbered partial sums, provided that the sequence of terms converges to zero, as must be the case for any convergent series. We conclude that the sequence of all partial sums converges, so the alternating series is convergent.

This leads to the Alternating Series Test: if the alternating series

\[
\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + \cdots,
\]

where \( b_n > 0 \) for \( n \geq 0 \), satisfies these conditions:

\[
b_{n+1} \leq b_n, \quad n \geq 0, \quad \text{and} \quad \lim_{n \to \infty} b_n = 0,
\]

then the series is convergent.

**Example** Consider the alternating series

\[
\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}.
\]

The terms in the series converge to zero as \( n \to \infty \). Furthermore, by differentiating the function

\[
f(x) = \frac{n}{n^2 + 1}
\]

with respect to \( x \), we can confirm that this function satisfies \( f'(1) = 0 \), and \( f'(x) < 0 \) for \( x > 1 \). It follows that this function is non-increasing for \( x \geq 1 \), and therefore the terms of the series are non-increasing. We conclude that the series passes the Alternating Series Test, and converges. \( \square \)

**Estimating Error in Alternating Series**

When computing the sum of a convergent series numerically, it is desirable to know how many terms are required in order to approximate the sum to within a given level of accuracy. For general
series, it is difficult to estimate the error incurred by truncating the series after a given number of terms, although for some series, a variant of the Integral Test may be used. For alternating series, however, it is particularly simple to estimate this error, if the series satisfies the Alternating Series Test.

Consider the general alternating series used to develop the Alternating Series Test. We established that the sequence of even-numbered partial sums is non-increasing. Similarly, the sequence of odd-numbered partial sums is non-decreasing. Since both sequences converge to the same limit, which is the sum $s$ of the series, it follows that $s$ lies between any two consecutive partial sums $s_n$ and $s_{n+1}$, for some $n \geq 0$. Therefore,

$$|s - s_n| \leq |s_{n+1} - s_n|.$$  

However, $s_{n+1} - s_n = b_{n+1}$, the next term in the series. We conclude that the error in the $n$th partial sum is bounded above by $b_{n+1}$.

We have just proved the **Alternating Series Estimation Theorem**: If an alternating series

$$\sum_{n=0}^{\infty} (-1)^n b_n,$$

where $b_n > 0$ for $n \geq 0$, satisfies these conditions:

$$b_{n+1} \leq b_n, \quad n \geq 0, \quad \text{and} \quad \lim_{n \to \infty} b_n = 0,$$

then

$$|s - s_n| \leq b_{n+1},$$

where $s$ is the sum of the series and $s_n$ is the $n$th partial sum.

**Example** Consider the convergent alternating series

$$\sum_{n=0}^{\infty} \left( -\frac{1}{3} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n}.$$  

We wish to approximate the sum $s$ of this series with a partial sum $s_n$ that includes enough terms so that $|s - s_n| \leq 0.001$. By the Alternating Series Estimation Theorem, we must choose $n$ so that $1/3^{n+1} \leq 0.001$. Rearranging, we obtain the condition $1000 \leq 3^{n+1}$, or, by taking the natural logarithm of both sides,

$$n \geq \frac{\ln 1000}{\ln 3} - 1 = \frac{3 \ln 10}{\ln 3} - 1 \approx 5.29.$$  

Therefore, we must use the first 7 terms (from $n = 0$ to $n = 6$) to approximate the sum.

In this case, we can confirm that using this many terms is necessary and sufficient, since we can compute the sum of the series. This is a geometric series with $a = 1$ and $r = -1/3$, so the sum is $1/(1 - (-1/3)) = 3/4$. Using the first 7 terms yields an error of approximately 0.000343, while using only the first 6 terms yields an error of approximately 0.00103. \(\square\)
Summary

- The Alternating Series Test states that an alternating series is convergent if its terms are non-increasing in magnitude, and converge to zero.

- If an alternating series passes the Alternating Series Test, then any partial sum of the series deviates from the overall sum by no more than the next term in the series.