1. Use Lagrange multipliers to find the point on the circle \( x^2 + y^2 = 4 \) closest to the point \((1, 5)\).

**Solution** We have \( f(x, y) = (x - 1)^2 + (y - 5)^2 \), the square of the distance from \((x, y)\) to \((1, 5)\), as the objective function that is to be minimized. The constraints are given by 
\[ g(x, y) = x^2 + y^2 - 4 = 0. \]
We then have 
\[ f_x = \lambda g_x, \quad f_y = \lambda g_y, \]
where \( \lambda \) is the Lagrange multiplier. This gives 
\[ 2(x - 1) = \lambda 2x, \quad 2(y - 5) = \lambda 2y, \quad x^2 + y^2 = 4. \]
From the first two equations, we obtain two distinct expressions for \( \lambda \). Equating them yields 
\[ \frac{x - 1}{x} = \frac{y - 5}{y} \]
which simplifies to \( y = 5x \). Substituting this into the constraint yields \( 26x^2 = 4 \), or \( x = \pm 0.39223 \). We then have the points \((x, y) = \pm (0.39223, 1.96116)\). Substituting these into \( f(x, y) \) gives the minimizer \((0.39223, 1.96116)\).

2. Evaluate the iterated integral 
\[ \int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x \, dz \, dy \, dx. \]

**Solution** We have 
\[ \int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x \, dz \, dy \, dx = \int_0^\pi \sin x \, dx \int_0^1 y \int_0^{\sqrt{1-y^2}} \, dz \, dy \]
\[ = -\cos x \bigg|_0^\pi \int_0^1 y \sqrt{1-y^2} \, dy \]
\[ = (0-(-1)) \int_0^1 y \sqrt{1-y^2} \, dy \]
\[ = \frac{2}{3} \int_0^1 u^{1/2} \, du, \quad u = 1 - y^2, \quad du = -2y \, dy \]
\[ = \frac{2}{3} \left[ \frac{1}{3} u^{3/2} \right]_0^1 \]
\[ = \frac{2}{3}. \]

3. Evaluate 
\[ \int \int_D \frac{y}{1+x^2} \, dA, \]
where \( D \) is the region bounded by \( y = \sqrt{x}, \ y = 0 \) and \( x = 1 \).
Solution We have

\[
\int \int_D \frac{y}{1 + x^2} \, dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1 + x^2} \, dy \, dx
\]

\[
= \int_0^1 \frac{1}{1 + x^2} \int_0^{\sqrt{x}} y \, dy \, dx
\]

\[
= \int_0^1 \frac{1}{1 + x^2} \frac{y^2}{2} \bigg|_0^{\sqrt{x}} \, dx
\]

\[
= \frac{1}{2} \int_0^1 \frac{x}{1 + x^2} \, dx
\]

\[
= \frac{1}{4} \int_1^2 \frac{1}{u} \, du, \quad u = 1 + x^2, \quad du = 2x \, dx
\]

\[
= \frac{1}{4} \ln |u|^2_1
\]

\[
= \frac{1}{4} \ln 2.
\]

4. Evaluate

\[
\int \int_D (x^2 + y^2)^{3/2} \, dA,
\]

where \( D \) is the region in the first quadrant bounded by the lines \( y = 0 \) and \( y = \sqrt{3}x \), and the circle \( x^2 + y^2 = 9 \).

Solution Let \( f(x, y) = (x^2 + y^2)^{3/2} \). Converting to polar coordinates, we obtain

\[
\int \int_D (x^2 + y^2)^{3/2} \, dA = \int_0^{\pi/3} \int_0^3 f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta
\]

\[
= \int_0^{\pi/3} \int_0^3 (r^2)^{3/2} \, r \, dr \, d\theta
\]

\[
= \int_0^{\pi/3} \int_0^3 r^4 \, dr \, d\theta
\]

\[
= \int_0^{\pi/3} d\theta \int_0^3 r^4 \, dr
\]

\[
= \frac{\pi}{3} \left[ \frac{r^5}{5} \right]_0^3
\]

\[
= \frac{81\pi}{5}.
\]

The upper limit of \( \theta, \pi/3 \), is obtained using the fact that the slope of the line \( y = \sqrt{3}x, \sqrt{3} \), is equal to \( \tan \frac{\pi}{3} \).

5. Evaluate

\[
\int \int \int_H z^2 \sqrt{x^2 + y^2 + z^2} \, dV,
\]

where \( H \) is the solid hemisphere that lies above the \( xy \)-plane and has center at the origin with radius 1.
\textbf{Solution} Using spherical coordinates, we obtain

\[
\int \int \int_H z^2 \sqrt{x^2 + y^2 + z^2} \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos \phi)^2 \rho (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \int_0^1 \rho^5 \, d\rho
\]

\[
= 2\pi \int_0^1 u^2 \, du \left(\frac{\rho^6}{6}\right)^1_0, \quad u = \cos \phi, \quad du = -\sin \phi \, d\phi
\]

\[
= 2\pi \left(\frac{u^3}{3}\right)^1_0 \frac{1}{6}
\]

\[
= \frac{\pi}{9}.
\]

6. Find the volume of the solid bounded by the cylinder \(x^2 + y^2 = 4\) and the planes \(z = 0\) and \(y + z = 3\).

\textbf{Solution} Using cylindrical coordinates, we obtain

\[
V = \int_0^{2\pi} \int_0^2 \int_0^{3-r \sin \theta} r \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 (r(3 - r \sin \theta)) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left(3r^2 - \frac{r^3 \sin \theta}{3}\right)^2 \, d\theta
\]

\[
= \int_0^{2\pi} \left(6r^2 - \frac{8}{3} \sin \theta \right) \, d\theta
\]

\[
= \left(6r^2 + \frac{8}{3} \cos \theta \right)^{2\pi}_0
\]

\[
= 12\pi.
\]

7. Use the transformation \(u = x - y, \quad v = x + y\) to evaluate

\[
\int \int_R \frac{x - y}{x + y} \, dA,
\]

where \(R\) is the square with vertices (0, 2), (1, 1), (2, 2), and (1, 3).

\textbf{Solution} Solving the above equations \(x\) and \(y\), we obtain

\[
x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(v - u).
\]

This yields

\[
\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| x_u y_v - y_u x_v \right| = \left| \frac{1}{2} \frac{1}{2} - \left(\frac{-1}{2}\right) \frac{1}{2} \right| = \frac{1}{2}.
\]
Substituting the vertices of the square into the change of variable yields the transformed vertices in $uv$-space: $(-2, 2), (0, 2), (0, 4), \text{and} (-2, 4)$. This yields the integral

$$\int \int_R \frac{x - y}{x + y} \, dA = \int_2^4 \int_{-2}^0 \frac{u}{v} \frac{\partial(x, y)}{\partial(u, v)} \, du \, dv$$

$$= \frac{1}{2} \int_2^4 \frac{1}{v} \, dv \int_{-2}^0 \frac{u}{2} \, du$$

$$= \frac{1}{2} \ln |v|^2 \bigg|_2^0$$

$$= \frac{1}{2} (\ln 4 - \ln 2) (-2)$$

$$= -\ln 2.$$

8. Evaluate the line integral

$$\int_C \sqrt{xy} \, dx + e^y \, dy + xz \, dz,$$

where $C$ is given by $r(t) = (t^4, t^2, t^3)$, $0 \leq t \leq 1$.

**Solution** From $r'(t) = (4t^3, 2t, 3t^2)$, we obtain

$$\int_C \sqrt{xy} \, dx + e^y \, dy + xz \, dz = \int_0^1 \sqrt{x(t)y(t)} \frac{dx}{dt} + e^{y(t)} \frac{dy}{dt} + x(t)z(t) \frac{dz}{dt} \, dt$$

$$= \int_0^1 t^3 (4t^3) + e^{t^2} (2t) + t^7 (3t^2) \, dt$$

$$= \int_0^1 4t^6 + 2te^{t^2} + 3t^9 \, dt$$

$$= \left( \frac{4t^7}{7} + e^{t^2} + \frac{3t^{10}}{10} \right) \bigg|_0^1$$

$$= \left( \frac{4}{7} + e + \frac{3}{10} \right) - 1.$$

9. Show that the vector field

$$\mathbf{F}(x, y, z) = (e^y, xe^y + e^z, ye^z)$$

is conservative, and use this fact to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$.

**Solution** A vector field $\mathbf{F} = (P, Q, R)$ is conservative if

$$R_y = Q_z, \quad P_z = R_x, \quad Q_x = P_y.$$

From

$$R_y = e^z = Q_z, \quad P_z = 0 = R_x, \quad Q_x = e^y = P_y,$$

we find that $\mathbf{F}$ is in fact conservative. To evaluate the line integral efficiently, we need to find a function $f$ such that $\nabla f = \mathbf{F}$. To that end, we obtain

$$f(x, y, z) = \int P \, dx = xe^y + g(y, z).$$
The requirement that \( f_y = Q \) yields the equation
\[
g_y(y, z) = Q(x, y, z) - (x e^y) y = e^z.
\]
Solving the equation for \( g_y \) yields
\[
g(y, z) = ye^z + h(z).
\]
The requirement that \( f_z = R \) yields the equation
\[
h'(z) = R(x, y, z) - (x e^y + ye^z) z = 0.
\]
It follows that \( h(z) = K \) where \( K \) is an arbitrary constant, and therefore
\[
f(x, y, z) = xe^y + ye^z + K.
\]
From the Fundamental Theorem of Line Integrals, we obtain
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}
\]
\[
= f(4, 0, 3) - f(0, 2, 0)
\]
\[
= (4e^0 + 0e^3) - (0e^2 + 2e^0)
\]
\[
= 2.
\]

10. Use Green’s Theorem to evaluate
\[
\int_C x^2 y\,dx - y^2\,dy,
\]
where \( C \) is the circle \( x^2 + y^2 = 4 \) with counterclockwise orientation.

**Solution** Let \( D = \{ (x, y) \mid x^2 + y^2 \leq 4 \} \) be the interior of \( C \). Then, by Green’s Theorem,
\[
\int_C x^2 y\,dx - y^2\,dy = \int \int_D (-xy^2)\,x - (x^2 y)\,y\,dA = \int \int_D -y^2 - x^2\,dA.
\]
Converting to polar coordinates, we obtain
\[
\int_C x^2 y\,dx - y^2\,dy = \int_0^{2\pi} \int_0^2 (-r^3) r\,dr\,d\theta
\]
\[
\quad = \int_0^{2\pi} d\theta \int_0^2 -r^4\,dr
\]
\[
\quad = 2\pi \left( -\frac{r^4}{4} \right) \bigg|_0^2
\]
\[
\quad = -8\pi.
\]

11. If \( f(x, y, z) \) and \( g(x, y, z) \) are twice differentiable functions, show that
\[
\nabla^2 (fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g,
\]
where \( \nabla^2 f = \nabla \cdot (\nabla f) \).
Solution We have
\[
\nabla^2 (fg) = \nabla \cdot (\nabla (fg))
= \nabla \cdot \langle (fg)_x, (fg)_y, (fg)_z \rangle
= \nabla \cdot \langle f_xg + fg_x + f_yg + fg_y, f_zg + fg_z \rangle
= (f_xg + fg_x)_x + (f_yg + fg_y)_y + (f_zg + fg_z)_z
= f_{xx}g + 2f_xg_x + f_{yy}g + 2f_yg_y + f_{yy}g_{zz} + f_{zz}g + f_{gg}z
= f\nabla \cdot (\nabla g) + g\nabla \cdot (\nabla f) + 2\nabla f \cdot \nabla g.
\]

In the last steps, we have used the fact that \(\nabla \cdot (\nabla f) = f_{xx} + f_{xy} + f_{yy}.\)

12. Evaluate the surface integral
\[
\int \int_S (x^2z + y^2z) \, dS,
\]
where \(S\) is the part of the plane \(z = 4 + x + y\) that lies inside the cylinder \(x^2 + y^2 = 4.\)

Solution We use the parametric equations
\[
x = u \cos v, \quad y = u \sin v, \quad z = 4 + u \cos v + u \sin v, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi,
\]
for which
\[
\mathbf{r}_u = \langle \cos v, \sin v, \cos v + \sin v \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, -u \sin v + u \cos v \rangle,
\]
\[
\mathbf{r}_u \times \mathbf{r}_v = \langle -u, -u, u \rangle, \quad ||\mathbf{r}_u \times \mathbf{r}_v|| = \sqrt{3}|u|.
\]
This yields
\[
\int \int_S (x^2z + y^2z) \, dS = \int_0^{2\pi} \int_0^2 [(u \cos v)^2 + (u \sin v)^2](4 + u \cos v + u \sin v)\sqrt{3}u \, du \, dv
= \sqrt{3} \int_0^{2\pi} \int_0^2 4u^3 + u^4 \cos v + u^4 \sin v \, du \, dv
= \sqrt{3} \int_0^{2\pi} \left[ u^4 + \frac{u^5}{5} \cos v \right]^2 \, dv
= \sqrt{3} \int_0^{2\pi} \left[ 16 + \frac{32}{5} (\cos v + \sin v) \right] \, dv
= 16\sqrt{3}(2\pi) + \frac{32}{5} (\sin v - \cos v)^2 \biggr|_0^{2\pi}
= 32\sqrt{3}\pi.
\]

13. Evaluate the surface integral
\[
\int \int_S \mathbf{F} \cdot d\mathbf{S},\]
where
\[
\mathbf{F}(x, y, z) = \langle xz, -2y, 3x \rangle
\]
and $S$ is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation.

**Solution** We use spherical coordinates

$$
x = 2 \sin \phi \cos \theta, \quad y = 2 \sin \phi \sin \theta, \quad z = 2 \cos \phi, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi,
$$

which yields

$$
r_\phi = (2 \cos \phi \cos \theta, 2 \cos \phi \sin \theta, -2 \sin \phi), \quad r_\theta = (-2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 0),
$$

$$
r_\phi \times r_\theta = 4 \sin \phi \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle,
$$

which has outward orientation since $4 \sin \phi \geq 0$ for $0 \leq \phi \leq \pi$. We then have

$$
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \frac{2 \sin \phi (2 \cos \theta \cos \phi, -2 \sin \theta, 3 \cos \theta) \cdot 4 \sin \phi (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)}{d\phi} d\theta
$$

$$
= 8 \int_0^{2\pi} \int_0^\pi \sin^2 \phi (2 \cos^2 \theta \sin \phi \cos \phi - 2 \sin^2 \theta \sin \phi + 3 \cos \theta \cos \phi) d\phi d\theta
$$

$$
= 16 \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi \cos \phi d\phi - 16 \int_0^{2\pi} \sin^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi +
$$

$$
24 \int_0^{2\pi} \cos \theta d\theta \int_0^\pi \sin^2 \phi \cos \phi d\phi
$$

$$
= 8 \int_0^{2\pi} 1 + \cos 2\theta d\theta \int_0^\pi u^3 du - \int \int \mathbf{F} \cdot d\mathbf{S} = 8 \int_0^{2\pi} \sin \phi \cos \phi d\phi
$$

$$
= 8 \int_0^{2\pi} \sin \phi \cos \phi d\phi = 8 \int_0^{2\pi} \sin \phi \cos \phi d\phi
$$

$$
= 16 \pi \left[ \frac{\sin^4 \phi}{4} \right]_0^\pi + 16 \pi \int_0^1 (1 - v^2) dv \quad (v = \cos \phi, dv = -\sin \phi d\phi)
$$

$$
= 16 \pi \left( \cos \phi - \frac{\cos^3 \phi}{3} \right)_0^\pi
$$

$$
= 16 \pi \left( -2 + \frac{2}{3} \right)
$$

$$
= \frac{64 \pi}{3}.
$$

Here, we have used the trigonometric identities

$$
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}.
$$

14. Use Stokes’ Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$ and $C$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented counterclockwise as viewed from above. **Hint:** to obtain an equation for the surface enclosed by $C$, compute the equation of a plane containing the vertices.

**Solution** Using the given vertices, it can be determined that $S$ is contained within the plane $x + y + z = 1$. We therefore describe $S$ using the parametric equations

$$
x = u, \quad y = v, \quad z = 1 - u - v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u,
which yields \( \mathbf{r}_u = \langle 1, 0, -1 \rangle, \ \mathbf{r}_v = \langle 0, 1, -1 \rangle, \) and the normal vector
\[
\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 1 \rangle,
\]
which is consistent with the counterclockwise orientation of \( C \) (that is, when traversing \( C \) such that this normal vector, which points upward, is visible, then the region \( S \) is on the left). Applying Stokes’ Theorem yields
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_0^{1-u} \text{curl} \mathbf{F}(u, v) : (\mathbf{r}_u \times \mathbf{r}_v) \, dv \, du.
\]
From
\[
\text{curl} \mathbf{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)
\]
\[= ((xz)_y - (yz)_z, (xy)_z - (xz)_x, (yz)_x - (xy)_y)
\]
\[= (-y, -z, -x).
\]
We then have
\[
\text{curl} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (-y, -z, -x) \cdot (1, 1, 1) = -(x + y + z),
\]
and thus
\[
\text{curl} \mathbf{F}(u, v) : (\mathbf{r}_u \times \mathbf{r}_v) = -(u + v + 1 - u - v) = -1.
\]
We conclude that
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \int_0^{1-u} 1 \, dv \, du = -A(D),
\]
where \( D \) is the triangle \( \{ (u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1 - u \} \). This triangle has base and height 1, which yields
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{2}.
\]
15. Use the Divergence Theorem to evaluate the surface integral \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F}(x, y, z) = (x^3, y^3, z^3) \) and \( S \) is the surface of the solid \( E \) bounded by the cylinder \( x^2 + y^2 = 1 \) and the planes \( z = 0 \) and \( z = 2 \).

**Solution** Using cylindrical coordinates, we obtain
\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div} \mathbf{F} \, dV
\]
\[= \iiint_E [(x^3)_x + (y^3)_y + (z^3)_z] \, dV
\]
\[= \iiint_E 3(x^2 + y^2 + z^2) \, dV
\]
\[= \int_0^{2\pi} \int_0^1 \int_0^2 (r^2 + z^2) r \, dz \, dr \, d\theta
\]
\[= \int_0^{2\pi} \int_0^1 \left( r^3 z + r \frac{z^3}{3} \right)_0^2 \, dr \, d\theta
\]
\[ = 3 \int_0^{2\pi} d\theta \int_0^1 2r^3 + \frac{8}{3} r \, dr \]

\[ = 6\pi \left( \frac{r^4}{2} + \frac{4r^2}{3} \right) \bigg|_0^1 \]

\[ = 11\pi. \]