1. Solve the equation

\[
\frac{dy}{dt} + y = \frac{1}{1 + e^t}.
\]

**Solution** This is a linear equation of the form

\[
\frac{dy}{dt} + p(t)y = q(t),
\]

with \( p(t) = 1 \) and \( q(t) = 1/(1 + e^t) \). Using the integrating factor

\[
\mu(t) = e^{\int p(t) \, dt} = e^{\int 1 \, dt} = e^t,
\]

we obtain the solution

\[
y(t) = \frac{1}{\mu(t)} \int \mu(t) q(t) \, dt
\]

\[
= \frac{1}{e^t} \int e^t \frac{1}{1 + e^t} \, dt
\]

\[
= e^{-t} \int \frac{1}{1 + u} \, du, \quad u = e^t, \quad du = e^t \, dt
\]

\[
= e^{-t} [\ln |1 + u| + C]
\]

\[
= e^{-t} [\ln |1 + e^t| + C]
\]

\[
= Ce^{-t} + e^{-t} \ln (1 + e^t),
\]

where \( C \) is an arbitrary constant.

2. Solve the equation

\[
\frac{dy}{dx} = \frac{1 + \cos x}{2 - \sin y}.
\]

**Solution** This is a separable equation, which can be rewritten in the form

\[(2 - \sin y) \, dy = (1 + \cos x) \, dx.\]

Integrating both sides yields

\[2y + \cos y = x + \sin x + C,\]

where \( C \) is an arbitrary constant.
3. Solve the equation
\[ \frac{dy}{dx} = \frac{x + y}{x - y}. \]

**Solution** This is a homogeneous equation, as it can be written in the form

\[ \frac{dy}{dx} = \frac{x + y}{x} \cdot \frac{1}{x - y} = \frac{1 + y/x}{1 - y/x}. \]

Using the change of variable \( v = y/x \), we obtain the new equation

\[ v + x \frac{dv}{dx} = \frac{1 + v}{1 - v}. \]

Rearranging, we obtain

\[ x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 - v} = \frac{1 + v^2}{1 - v}. \]

This equation is separable, and can be written in the form

\[ \frac{1 - v}{1 + v^2} dv = \frac{1}{x} dx. \]

Integrating both sides yields

\[ \tan^{-1} v - \frac{1}{2} \ln(1 + v^2) = \ln |x| + C, \]

where \( C \) is an arbitrary constant. Using the relation \( v = y/x \), we obtain

\[ \tan^{-1}(y/x) - \ln \sqrt{1 + y^2/x^2} = \ln |x| + C. \]

Using the properties of logarithms, we have

\[ \ln \sqrt{1 + y^2/x^2} = \ln \sqrt{\frac{x^2 + y^2}{x^2}} = \ln \sqrt{x^2 + y^2} - \ln \sqrt{x^2} = \ln \sqrt{x^2 + y^2} - \ln |x|, \]

which allows the equation for the solution to be simplified to

\[ \tan^{-1}(y/x) - \ln \sqrt{x^2 + y^2} = C. \]

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon (ick!) is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Write down a differential
equation for $Q(t)$, the amount of salt in the tank at time $t$ where $t$ is measured in minutes. Then solve the equation.

**Solution** The differential equation for $Q(t)$ has the form

$$\frac{dQ}{dt} = \text{inflow rate} - \text{outflow rate}.$$ 

The inflow rate is the product of the flow rate of the entering solution, 3 gal/min, and the concentration of salt in the entering solution, 1 lb/gal, which is 3 lb/min. The outflow rate is the product of the flow rate of the exiting solution, 2 gal/min, and the concentration of salt in the exiting solution, which is the ratio of the amount of salt, $Q(t)$, to the amount of water, which is $200 + t$. This is because the tank initially contains 200 gal of water, and with each minute 3 gallons flow in and 2 gallons flow out, causing a net increase of 1 gal/min.

It follows that the differential equation is

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200 + t},$$

with initial condition $Q(0) = 100$, since the tank initially contains 100 lb of salt in solution. This is a linear equation of the form

$$\frac{dy}{dt} + p(t)y = q(t),$$

where $p(t) = 2/(200 + t)$ and $q(t) = 3$. The integrating factor is

$$\mu(t) = e^{\int \frac{2}{200+t} \, dt} = e^{2 \ln |200+t|} = e^{\ln(200+t)^2} = (200 + t)^2.$$

Therefore the solution is

$$Q(t) = \frac{1}{\mu(t)} \int \mu(t)q(t) \, dt$$

$$= \frac{1}{(200 + t)^2} \int 3(200 + t)^2 \, dt$$

$$= \frac{1}{(200 + t)^2} [(200 + t)^3 + C]$$

where $C$ is an arbitrary constant. To determine the value of $C$, we use the initial condition $Q(0) = 100$, which yields the equation

$$100 = \frac{1}{200^2}[200^3 + C].$$

Rearranging yields

$$C = 100(200)^2 - 200^3 = 200^2(100 - 200) = -100(200)^2.$$
We conclude that the solution is
\[
Q(t) = \frac{1}{(200 + t)^2} \left[ (200 + t)^3 - 100(200)^2 \right] = 200 + t - \frac{100(200)^2}{(200 + t)^2}.
\]

5. For the equation
\[
\frac{dy}{dt} = y^2(1 - y^2),
\]
determine the equilibrium points, and classify each one as asymptotically stable, unstable, or semistable. Draw the phase line. You do not need to solve the equation.

**Solutions** The equilibrium points are \( y = 0, 1, -1 \). When \( y < -1 \), \( dy/dt < 0 \), and when \(-1 < y < 0\), \( dy/dt > 0 \), so \( y = -1 \) is unstable, as \( dy/dt \) is trending away from \( y = -1 \) on either side. When \( 0 < y < 1 \), \( dy/dt > 0 \), so \( y = 0 \) is semi-stable, as \( dy/dt > 0 \) on both sides of \( y = 0 \). Finally, when \( y > 1 \), \( dy/dt < 0 \), so \( y = 1 \) is stable, as \( dy/dt \) is trending toward \( y = 1 \) on either side.

6. Solve the equation
\[
(x^2 + y) \, dx + (x + e^y) \, dy = 0.
\]

**Solution** This is an equation of the form
\[
M(x, y) \, dx + N(x, y) \, dy = 0,
\]
where \( M(x, y) = x^2 + y \) and \( N(x, y) = x + e^y \). We check whether this equation is exact. From
\[
M_y = 1, \quad N_x = 1,
\]
we find that it is exact.

To solve the equation, we compute
\[
M_1(x, y) = \int M(x, y) \, dx = \int (x^2 + y) \, dx = \frac{x^3}{3} + xy,
\]
and
\[
N(x, y) - \frac{\partial}{\partial y} M_1(x, y) = x + e^y - \left( \frac{x^3}{3} + xy \right)_y = x + e^y - x = e^y.
\]
Therefore, the solution \( y \) satisfies the equation
\[
M_1(x, y) + \int N(x, y) - \frac{\partial}{\partial y} M_1(x, y) \, dy = 0,
\]
which is
\[
\frac{x^3}{3} + xy + \int e^y \, dy = \frac{x^3}{3} + xy + e^y + C = 0,
\]
where \( C \) is an arbitrary constant.
7. Solve the equation

\[(2y + 3x)\, dx + x\, dy = 0.\]

**Solution** This is an equation of the form

\[M(x, y)\, dx + N(x, y)\, dy = 0,\]

where \(M(x, y) = 2y + 3x\) and \(N(x, y) = x\). We check whether this equation is exact. From

\[M_y = 2, \quad N_x = 1,\]

we find that it is not exact.

We then compute

\[
\frac{M_y - N_x}{N} = \frac{2 - 1}{x} = \frac{1}{x}, \quad \frac{N_x - M_y}{M} = \frac{1 - 2}{2y + 3x} = -\frac{1}{2y + 3x}.
\]

As the first expression is a function of only \(x\), our integrating factor is

\[
\mu(x) = e^{\int \frac{M_y - N_x}{N} \, dx} = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x.
\]

Then, our scaled equation

\[(2xy + 3x^2)\, dx + x^2\, dy = 0,\]

with \(M(x, y) = 2xy + 3x^2\) and \(N(x, y) = x^2\), is exact, as \(M_y = N_x = 2x\).

To solve the equation, we compute

\[
M_1(x, y) = \int M(x, y)\, dx = \int 2xy + 3x^2\, dx = x^2y + x^3,
\]

and

\[
N(x, y) - \frac{\partial}{\partial y} M_1(x, y) = x^2 - (x^2y + x^3)_y = x^2 - x^2 = 0.
\]

Therefore, the solution \(y\) satisfies the equation

\[M_1(x, y) + \int N(x, y) - \frac{\partial}{\partial y} M_1(x, y)\, dy = 0,\]

which is

\[x^2y + x^3 + \int 0\, dy = x^2y + x^3 + C = 0,\]

where \(C\) is an arbitrary constant.