1. The indicated function \( y_1(x) \) is a solution of the given differential equation. Use reduction of order or formula (5) in Section 4.2,

\[
y_2(x) = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx
\]
as instructed, to find a second solution \( y_2(x) \).

\[ x^2 y'' + 2xy' - 6y = 0; \quad y_1 = x^2. \]

**Solution** First, we divide both sides of the differential equation by the coefficient of \( y'' \), which is \( x^2 \), to obtain

\[ y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0. \]

This yields \( P(x) = 2/x \). We then have

\[
y_2(x) = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx
\]

\[
= x^2 \int \frac{e^{-\int \frac{2}{x} \, dx}}{x^4} \, dx
\]

\[
= x^2 \int \frac{e^{-2 \ln x}}{x^4} \, dx
\]

\[
= x^2 \int \frac{e^{\ln x^{-2}}}{x^4} \, dx
\]

\[
= x^2 \int \frac{x^{-2}}{x^4} \, dx
\]

\[
= x^2 \int x^{-6} \, dx
\]

\[
= x^2 \frac{x^{-5}}{-5}
\]

\[
= -\frac{1}{5} x^{-3}.
\]

Since constant factors can be neglected when constructing a set of fundamental solutions, we can conclude that a second solution is \( y_2(x) = 1/x^3 \).
2. Find the general solution of the given second-order differential equation.

\[ 6y'' + y' = 0 \]

**Solution** The characteristic equation is \( 6r^2 + r = 0 \), which factors into \( r(6r + 1) = 0 \). The solutions are \( r = -1/6 \) and \( r = 0 \). Since these roots are real and distinct, we conclude that the general solution is

\[ y(x) = C_1e^{-x/6} + C_2x^0 = C_1e^{-x/6} + C_2. \]

3. Find the general solution of the given second-order differential equation.

\[ y'' + 4y' + 4y = 0. \]

**Solution** The characteristic equation is \( r^2 + 4r + 4 = 0 \), which factors into \( (r + 2)^2 = 0 \). The only solution is \( r = -2 \). Since we have a repeated root, we conclude that the general solution is

\[ y(x) = C_1e^{-2x} + C_2xe^{-2x}. \]

4. Find the general solution of the given second-order differential equation.

\[ y'' - 6y' + 10 = 0. \]

**Solution** The characteristic equation is \( r^2 - 6r + 10 = 0 \). This does not factor nicely, so we use the quadratic formula to obtain the roots

\[ r = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(10)}}{2} = 3 \pm \frac{-4}{2} = 3 \pm i. \]

The roots are complex, which yields the general solution

\[ y(x) = C_1e^{3x} \cos x + C_2e^{3x} \sin x. \]

5. Solve the given initial-value problem.

\[ 5y'' + y' = -4x, \quad y(0) = 0, \quad y'(0) = -15. \]

**Solution** First, we obtain the general solution of the homogeneous equation

\[ 5y'' + y' = 0. \]
The characteristic equation is $5r^2 + r = 0$, which factors into $r(5r + 1) = 0$. The solutions are $r = -1/5$ and $r = 0$. Since these roots are real and distinct, we have the general solution

$$y_h(x) = C_1e^{-x/5} + C_2x^0 = C_1e^{-x/5} + C_2.$$

Next, since the differential equation has constant coefficients, we use the Method of Undetermined Coefficients to obtain a particular solution. To determine the form of the particular solution $y_p(x)$, we note that the right side is a polynomial of degree 1, but 0 is also a root of the characteristic equation. It follows that $y_p(x)$ has the form

$$y_p(x) = x(A_1x + A_0) = A_1x^2 + A_0x.$$

To substitute this form into the differential equation, we first obtain

$$y'_p(x) = 2A_1x + A_0, \quad y''_p(x) = 2A_1.$$

We then substitute, which yields

$$5(2A_1) + (2A_1x + A_0) = 10A_1 + 2A_1x + A_0 = -4x.$$

Matching terms with like powers of $x$, we obtain the equations

$$10A_1 + A_0 = 0, \quad 2A_1x = -4x.$$

These equation have the solutions $A_1 = -2, A_0 = 20$. We therefore have the particular solution

$$y_p(x) = -2x^2 + 20x.$$

The general solution of the differential equation is

$$y(x) = y_h(x) + y_p(x) = C_1e^{-x/5} + C_2 - 2x^2 + 20x.$$

To obtain the values of $C_1$ and $C_2$, we substitute $x = 0$ into $y(x)$ and its derivative

$$y'(x) = -\frac{1}{5}C_1e^{-x/5} - 4x + 20$$

and use the initial conditions to obtain the equations

$$y(0) = C_1 + C_2 = 0, \quad y'(0) = -\frac{1}{5}C_1 + 20 = -15.$$

The second equation yields $C_1 = 175$, and then the first equation yields $C_2 = -175$. We conclude that the solution of the initial value problem is

$$y(x) = 175e^{-x/5} - 175 - 2x^2 + 20x.$$
6. Solve the differential equation by variation of parameters.

\[ 3y'' - 6y' + 6y = e^x \sec x. \]

**Solution** First, we obtain the general solution of the homogeneous equation

\[ 3y'' - 6y' + 6y = 0. \]

The characteristic equation is \( 3r^2 - 6r + 6 = 0 \). This does not factor nicely, so we use the quadratic formula to obtain the roots

\[ r = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(6)}}{2(3)} = 1 \pm \frac{\sqrt{-36}}{6} = 1 \pm i. \]

The roots are complex, which yields the general solution

\[ y_h(x) = C_1 y_1(x) + C_2 y_2(x), \quad y_1(x) = e^x \cos x, \quad y_2(x) = e^x \sin x. \]

Now, we use the method of Variation of Parameters to obtain a particular solution. First, we need to rewrite the equation so that the coefficient of \( y'' \) is 1. We then work with

\[ y'' - 2y' + 2y = \frac{1}{3} e^x \sec x. \]

Then, we compute the Wronskian

\[
W(y_1, y_2) = y_1 y'_2 - y_2 y'_1 \\
= (e^x \cos x)(e^x \sin x)' - (e^x \sin x)(e^x \cos x)' \\
= (e^x \cos x)(e^x \sin x + e^x \cos x) - (e^x \sin x)(e^x \cos x - e^x \sin x) \\
= e^{2x} \cos x \sin x + e^{2x} \cos^2 x - e^{2x} \sin x \cos x + e^{2x} \sin^2 x \\
= e^{2x}.
\]

Then, the particular solution has the form

\[ y_p(x) = w_1(x)y_1(x) + w_2(x)y_2(x), \]

where

\[ w_1(x) = - \int \frac{y_2(x)\frac{1}{3}e^x \sec x}{W(y_1, y_2)} \, dx, \quad w_2(x) = \int \frac{y_1(x)\frac{1}{3}e^x \sec x}{W(y_1, y_2)} \, dx. \]

Applying these formulas with the previously obtained \( y_1(x) \) and \( y_2(x) \) yields

\[ w_1(x) = -\frac{1}{3} \int \frac{(e^x \sin x)e^x \sec x}{e^{2x}} \, dx = -\frac{1}{3} \int \frac{\sin x}{\cos x} \, dx = \frac{1}{3} \ln(|\cos x|), \]

and

\[ w_2(x) = \frac{1}{3} \int \frac{(e^x \cos x)e^x \sec x}{e^{2x}} \, dx = \frac{1}{3} \int \frac{\cos x}{\cos x} \, dx = \frac{1}{3} \ln(|\cos x|). \]
\[
w_2(x) = \frac{1}{3} \int \left( e^x \cos x \right) e^x \sec x \ dx = \frac{1}{3} \int 1 \ dx = \frac{1}{3} x.
\]

We then have the particular solution

\[
y_p(x) = w_1(x) y_1(x) + w_2(x) y_2(x) = \frac{1}{3} \ln(\cos x) e^x \cos x + \frac{1}{3} x e^x \sin x.
\]

We conclude that the general solution is

\[
y(x) = y_h(x) + y_p(x) = C_1 e^x \cos x + C_2 e^x \sin x + \frac{1}{3} \ln(\cos x) e^x \cos x + \frac{1}{3} x e^x \sin x.
\]

7. A force of 960 newtons stretches a spring 4 meters. A mass of 60 kilograms is attached to
the end of the spring and is initially released from the equilibrium position with an upward
velocity of 6 m/s. Find the equation of motion.

**Solution** Since there is no damping and no external force, the position of the mass satisfies
the differential equation

\[
m x'' + k x = 0
\]

where \( m = 60 \), the mass. The spring constant \( k \) satisfies the equation \( F = kL \), where \( F \) is
the force, which is 960 newtons, and \( L \) is the length of the spring resulting from that force,
4 meters. This yields a spring constant of \( k = 240 \). Therefore the differential equation is

\[
60x'' + 240x = 0,
\]

which simplifies to

\[
x'' + 4x = 0.
\]

Because the mass is released from the equilibrium position, one initial condition is \( x(0) = 0 \).
The upward velocity of 6 m/s yields the second initial condition \( x'(0) = -6 \), since the
downward direction is positive.

The characteristic equation is \( r^2 + 4 = 0 \), which has roots \( r = \pm 2i \). Therefore the general
solution is

\[
x(t) = C_1 \cos(2t) + C_2 \sin(2t).
\]

To obtain the values of \( C_1 \) and \( C_2 \), we substitute \( t = 0 \) into \( x(t) \) and

\[
x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t).
\]

This yields the equations

\[
x(0) = C_1 = 0, \quad x'(0) = 2C_2 = -6.
\]

The solutions of these equations are \( C_1 = 0 \) and \( C_2 = -3 \). We conclude that the equation of
motion is

\[
x(t) = -3 \sin(2t).
\]