1. Solve the initial value problem

\[ y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1. \]

**Solution** The characteristic equation is

\[ \lambda^2 + 4\lambda + 3 = 0, \]

which factors into

\[ (\lambda + 1)(\lambda + 3) = 0. \]

Therefore, the roots are \( \lambda_1 = -1 \) and \( \lambda_2 = -3 \). Since the roots are real and distinct, the general solution is

\[ y(t) = c_1 e^{-t} + c_2 e^{-3t}, \]

which has the derivative

\[ y'(t) = -c_1 e^{-t} - 3c_2 e^{-3t}. \]

From the initial conditions, we obtain the equations

\[ y(0) = c_1 + c_2 = 2, \quad y'(0) = -c_1 - 3c_2 = -1. \]

By adding these two equations, we obtain \(-2c_2 = 1\), and therefore \(c_2 = -1/2\). Substituting this value into either equation yields \(c_1 = 5/2\). We conclude that the solution is

\[ y(t) = \frac{5}{2} e^{-t} - \frac{1}{2} e^{-3t}. \]

2. Solve the initial value problem

\[ y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

**Solution** The characteristic equation is

\[ \lambda^2 + 4\lambda + 5 = 0, \]

which has roots

\[ \lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm \sqrt{-1} = -2 \pm i. \]
Since the roots are complex, the general solution is
\[ y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t, \]
which has the derivative
\[ y'(t) = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t. \]

From the initial conditions, we obtain the equations
\[ y(0) = c_1 = 1, \quad y'(0) = -2c_1 + c_2 = 0. \]

It follows that \( c_1 = 1 \) and \( c_2 = 2 \). We conclude that the solution is
\[ y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t. \]

3. Use reduction of order to find a second solution of the equation
\[ xy'' - y' + 4x^3 y = 0, \quad y_1(x) = \sin(x^2). \]

**Solution** First, we divide both sides of the equation by the coefficient of \( x \) to obtain
\[ y'' - \frac{1}{x} y' + 4x^2 y = 0. \]

Then, this equation has the form
\[ y'' + p(x)y' + q(x)y = 0, \]
where \( p(x) = 1/x \). We assume that the second solution \( y_2(x) \) has the form
\[ y_2(x) = y_1(x)v(x). \]

Then \( v(x) \) is given by
\[
v(x) &= \int \frac{1}{|y_1(x)|^2} e^{-\int p(x) \, dx} \, dx \\
&= \int \frac{1}{|\sin(x^2)|^2} e^{-\frac{1}{x} \, dx} \\
&= \int \frac{1}{|\sin(x^2)|^2} e^{\ln x} \, dx \\
&= \int \frac{x}{|\sin(x^2)|^2} \, dx.
\]
\[
\frac{1}{2} \int \frac{1}{\sin^2 u} \, du, \quad u = x^2 \\
= \frac{1}{2} \int \csc^2 u \, du \\
= -\frac{1}{2} \cot u \\
= -\frac{1}{2} \cot(x^2).
\]

Neglecting the constant factor, we conclude that a second solution is
\[
y_2(x) = \sin(x^2) \cot(x^2) = \sin(x^2) \frac{\cos(x^2)}{\sin(x^2)} = \cos(x^2).
\]

4. Solve the initial value problem
\[
9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1.
\]

**Solution** The characteristic equation is
\[
9\lambda^2 - 12\lambda + 4 = 0,
\]
which has roots
\[
\lambda_{1,2} = \frac{12 \pm \sqrt{(-12)^2 - 4(9)(4)}}{2(9)} = \frac{2}{3}.
\]

Since this is a double root, the general solution is of the form
\[
y(t) = c_1 e^{2t/3} + c_2 te^{2t/3},
\]
which has the derivative
\[
y'(t) = \frac{2}{3} c_1 e^{2t/3} + c_2 e^{2t/3} + \frac{2}{3} c_2 t e^{2t/3}.
\]

From the initial conditions, we obtain the equations
\[
y(0) = c_1 = 2, \quad y'(0) = \frac{2}{3} c_1 + c_2 = -1.
\]

It follows that \(c_1 = 2\) and \(c_2 = -1 - 2(2)/3 = -7/3\). We conclude that the solution is
\[
y(t) = 2e^{2t/3} - \frac{7}{3} te^{2t/3}.
\]
5. Find the general solution of the differential equation

\[ y'' + 2y' + y = 2e^{-t}. \]

**Solution** The characteristic equation is

\[ \lambda^2 + 2\lambda + 1 = 0, \]

which factors into \((\lambda + 1)^2 = 0\). It follows that the roots are both equal to \(-1\), and therefore the general solution of the **homogeneous** equation is

\[ y_h(t) = c_1 e^{-t} + c_2 te^{-t}. \]

Now, we need to find a particular solution of the inhomogeneous equation. Since the equation has constant coefficients and the right-hand side \(g(t) = 2e^{-t}\) is of an appropriate form, we can use the method of undetermined coefficients. Specifically, \(g(t)\) is of the form \(P_n(t)e^{at}\). Therefore, the particular solution is of the form

\[ y_p(t) = t^s(A_0 + A_1 t + \cdots + A_n t^n)e^{-t}. \]

In this case, \(n = 0\), \(P_0(t) = 2\), and \(a = -1\). Since \(a\) is occurs twice as a root of the characteristic equation, \(s = 2\). Therefore, the particular solution has the form

\[ y_p(t) = t^2 A_0 e^{-t} \]

where \(A_0\) is an undetermined coefficient.

We now substitute this form of \(y_p(t)\) into the ODE. From

\[ y_p'(t) = A_0(2t - t^2)e^{-t}, \quad y_p''(t) = A_0(t^2 - 4t + 2)e^{-t}, \]

we obtain

\[ A_0(t^2 - 4t + 2)e^{-t} + 2[A_0(2t - t^2)e^{-t}] + A_0 t^2 e^{-t} = 2e^{-t}, \]

which simplifies to

\[ 2A_0 = 2. \]

It follows that \(A_0 = 1\) and that

\[ y_p(t) = t^2 e^{-t}. \]

We conclude that the general solution is

\[ y(t) = y_h(t) + y_p(t) = c_1 e^{-t} + c_2 te^{-t} + t^2 e^{-t}. \]
6. Find the general solution of the differential equation

\[ t^2y'' - 2ty' + 2y = 4t^2, \quad y_1(t) = t, \quad y_2(t) = t^2. \]

**Solution** Since this inhomogeneous equation does not have constant coefficients, we cannot use the method of undetermined coefficients. Instead, we can use variation of parameters. First, we divide both sides of the equation by the coefficient of \( t^2 \), to obtain

\[ y'' - \frac{2}{t} y' + \frac{2}{t^2} y = 4. \]

Then, compute the Wronskian of the homogeneous solutions \( y_1 \) and \( y_2 \), which is

\[ W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = t(t^2)' - t^2(t') = 2t^2 - t^2 = t^2. \]

Then, the particular solution is of the form

\[ y_p(t) = y_1(t)w_1(t) + y_2(t)w_2(t) \]

where

\begin{align*}
  w_1(t) &= \int \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} \, dt = -\int \frac{4t^2}{t^2} \, dt = -4 \int dt = -4t, \\
  w_2(t) &= \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} \, dt = 4 \int \frac{1}{t} \, dt = 4 \ln t.
\end{align*}

We conclude that a particular solution is

\[ y_p(t) = -4t^2 + 4t^2 \ln t, \]

and that the general solution is

\[ y(t) = y_h(t) + y_p(t) = c_1 + c_2t^2 + 4t^2 \ln t. \]

In the general solution, the \(-4t^2\) term has been dropped because it is a solution of the homogeneous equation.

7. A object with a mass of 100 g stretches a spring 5 cm. If the object is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position \( u \) of the object at any time \( t \). Find the frequency, period, and amplitude of the motion.

**Solution** Since there is no damping and no external force, the motion of the object is modeled by the equation

\[ mu'' + ku = 0, \]
where $u$ is measured in meters, $m = 100 \text{ g}$, and the spring constant $k$ satisfies the equation

$$kL - mg = 0,$$

where $L = 0.05 \text{ m}$ and $g = 9.8 \text{ m/s}^2$. Therefore,

$$k = \frac{mg}{L} = \frac{(100 \text{ g})(9.8 \text{ m/s}^2)}{0.05 \text{ m}} = 19,600 \text{ g/s}^2.$$

The initial conditions are

$$u(0) = 0, \quad u'(0) = 0.1,$$

since the object is set in motion from its equilibrium position with a downward velocity of 10 cm/s, which is 0.1 m/s.

Dividing both sides of the ODE by the coefficient of $u''$ yields

$$u'' + 196u = 0.$$

The characteristic equation is

$$\lambda^2 + 196 = 0,$$

which has complex roots $\lambda_{1,2} = \pm 14i$. It follows that the solution is of the form

$$u(t) = A \cos 14t + B \sin 14t,$$

which has a derivative of

$$u'(t) = -14A \sin 14t + 14B \cos t.$$

From the initial conditions, we obtain the equations

$$u(0) = A = 0, \quad u'(0) = 14B = 0.1,$$

which yields $A = 0$ and $B = 1/140$. Therefore, the solution is

$$u(t) = \frac{1}{140} \sin 14t.$$

The frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{19600}{100}} = \sqrt{196} = 14.$$

The period is

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{14} = \frac{\pi}{7},$$

and the amplitude is

$$R = \sqrt{A^2 + B^2} = \frac{1}{140}.$$