These notes correspond to Lesson 17 in the text.

The D’Alembert Solution of the Wave Equation

The solution of the Cauchy problem for the wave equation in one space dimension,

\[ u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty, \]

is known as d’Alembert’s solution. We will derive this solution in two different ways.

For the first approach, we introduce the change of variables

\[ \xi = x + ct, \quad \eta = x - ct. \]

Then we have

\[ u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, \]

\[ u_{xx} = (u_x)_\xi + (u_x)_\eta = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \]

\[ u_t = u_\xi t + u_\eta t = cu_\xi - cu_\eta, \]

\[ u_{tt} = c(u_t)_\xi - c(u_t)_\eta = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}. \]

Substituting these expressions for \( u_{xx} \) and \( u_{tt} \) into the wave equation yields the very simple PDE

\[ u_{\xi\eta} = 0. \]

By integrating with respect to \( \xi \), and then with respect to \( \eta \), we obtain the general solution

\[ u(\xi, \eta) = \Phi(\eta) + \psi(\xi) \]

where the functions \( \Phi(\eta) \) and \( \psi(\xi) \) are chosen so as to satisfy the initial conditions.

Substituting this expression into the initial conditions yields the equations

\[ \Phi(x) + \psi(x) = f(x), \]

\[ -c\Phi'(x) + cv'(x) = g(x). \]

This system of equations has the solutions

\[ \Phi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^x g(s) \, ds, \quad \psi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^x g(s) \, ds \]
for some $x_0$. It follows that
\[ u(x, t) = \Phi(x - ct) + \psi(x + ct) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]

For the second approach, we consider the IBVP
\[ u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < L, \]
\[ u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0. \]

We assume a solution of the form
\[ u(x, t) = X(x)T(t). \]

Separation of variables yields the ODEs
\[ X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad X(L) = 0, \]
\[ T'' + c^2 \lambda^2 T = 0. \]

The boundary value problem for $X$ has the solutions
\[ X_n(x) = \sin \left( \frac{n\pi x}{L} \right), \quad n = 1, 2, \ldots, \]
and therefore the ODE for $T$ has the general solutions
\[ T_n(t) = A_n \cos \left( \frac{n\pi c t}{L} \right) + B_n \sin \left( \frac{n\pi c t}{L} \right), \quad n = 1, 2, \ldots, \]
where $A_n$ and $B_n$ are constants.

It follows that the solution of our IBVP has the form
\[
\begin{align*}
    u(x, t) &= \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi c t}{L} \right) \sin \left( \frac{n\pi x}{L} \right) + B_n \sin \left( \frac{n\pi c t}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \\
    &= \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[ \sin \left( \frac{n\pi (x + ct)}{L} \right) + \sin \left( \frac{n\pi (x - ct)}{L} \right) \right] - \frac{1}{2} \sum_{n=1}^{\infty} B_n \left[ \cos \left( \frac{n\pi (x + ct)}{L} \right) - \cos \left( \frac{n\pi (x - ct)}{L} \right) \right].
\end{align*}
\]

Substituting this solution into the initial conditions yields
\[
\begin{align*}
    f(x) &= \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi x}{L} \right), \\
    g(x) &= c \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \sin \left( \frac{n\pi x}{L} \right).
\end{align*}
\]

Applying the Fundamental Theorem of Calculus “in reverse” to the cosine terms in the solution yields
\[ u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]