These notes correspond to Lesson 18 in the text.

More on the D’Alembert Solution

Earlier, we learned that the solution of the initial value problem
\[ u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \]
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty \]
is given by D’Alembert’s solution
\[ u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]

We now examine how this solution can be interpreted.

The Space-Time Interpretation of D’Alembert’s Solution

First, we consider the case of a zero initial velocity, which has initial conditions
\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad -\infty < x < \infty. \]
Then, the solution is
\[ u(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)]. \]

It follows that at any point \((x_0, t_0)\), the solution is equal to the average of the initial displacement \(u(x, 0) = f(x)\) at the two points obtained by backtracking along the lines
\[ x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0 \]
back to the \(x\)-axis.

For example, suppose the initial displacement is given by
\[ f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases}. \]

Then, \(u(x, t) = 1/2\) between the lines \(x + ct = \pm 1\), and between the lines \(x - ct = \pm 1\). Where these regions overlap, these values of \(u(x, t)\) are added, and the solution is equal to 1. Outside of these regions, \(u(x, t) = 0\).

Next, we consider the case of a zero initial displacement,
\[ u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty. \]
Then, the solution is
\[ u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]
That is, the solution at \((x_0, t_0)\) is obtained by integrating the initial velocity \(u_t(x, 0) = g(x)\) along the \(x\)-axis from \(x_0 - ct_0\) to \(x_0 + ct_0\).

Therefore, if the initial velocity is given by
\[
g(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & |x| \geq 1 \end{cases},
\]
then \(u(x, t) = (1 + x + ct)/(2c)\) between the lines \(x + ct = \pm 1\), and \(u(x, t) = (1 - x + ct)/(2c)\) between the lines \(x - ct = \pm 1\). Where these regions overlap, the solution is equal to \(t\). Between these two regions, the solution is equal to \(1/c\); everywhere else, it is equal to 0.

**Solution of the Semi-Infinite String via the D’Alembert Solution**

We now consider a vibrating semi-infinite string with a fixed end, modeled by the IBVP
\[
\begin{align*}
&u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \\
&u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < \infty, \\
&u(0, t) = 0, \quad t > 0.
\end{align*}
\]

As with the infinite string, using the change of variables
\[
\xi = x + ct, \quad \eta = x - ct,
\]
we obtain the much simpler PDE
\[
u_{\xi\eta} = 0,
\]
which has the general solution
\[
u(x, t) = \phi(\eta) + \psi(\xi) = \phi(x - ct) + \psi(x + ct).
\]

As before, we substitute this form of the solution into the initial conditions, and obtain
\[
\phi(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_{x_0}^{x - ct} g(s) \, ds, \quad \psi(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x_0}^{x + ct} g(s) \, ds.
\]

However, we can only evaluate \(f(x)\) and \(g(x)\) wherever \(x > 0\), which presents a problem when \(x - ct < 0\). To get around this, we apply the boundary condition to the form of \(u(x, t)\) to obtain
\[
u(0, t) = \phi(-ct) + \psi(ct) = 0,
\]
or
\[
\phi(-ct) = -\psi(ct).
\]

This yields
\[
\phi(x - ct) = -\psi(ct - x) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int_{x_0}^{ct - x} g(s) \, ds,
\]
and therefore
\[
u(x, t) = \psi(x + ct) - \psi(ct - x) = \frac{1}{2} [f(x + ct) - f(ct - x)] + \frac{1}{2c} \int_{ct - x}^{x + ct} g(s) \, ds, \quad 0 < x < ct.
\]

When \(x \geq ct\), we simply use D’Alembert’s solution as before,
\[
u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds, \quad x \geq ct.
\]

This solution exhibits reflection at the boundary \(x = 0\).