The Wave Equation in Two and Three Dimensions

Waves in Three Dimensions

Consider the IVP

$$u_{tt} = c^2 \nabla^2 u, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0,$$

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = \psi(x, y, z),$$

where $\nabla^2 u$ is the Laplacian

$$\nabla^2 u = \nabla \cdot \nabla u = u_{xx} + u_{yy} + u_{zz}.$$

Using the 3-D Fourier transform,

$$\hat{u}(\omega_1, \omega_2, \omega_3, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i(\omega_1 x + \omega_2 y + \omega_3 z)} u(x, y, z, t) \, dV,$$

we obtain the ODE

$$\hat{u}_{tt} = -c^2 (\omega_1^2 + \omega_2^2 + \omega_3^2) \hat{u},$$

with initial conditions

$$\hat{u}(\omega_1, \omega_2, \omega_3, 0) = 0, \quad \hat{u}_t(\omega_1, \omega_2, \omega_3, 0) = \hat{\psi}(\omega_1, \omega_2, \omega_3),$$

where $\hat{\psi}$ is the Fourier transform of $\psi$. This IVP has the solution

$$\hat{u}(\omega_1, \omega_2, \omega_3, t) = \frac{1}{c ||\vec{\omega}||} \hat{\psi}(\omega_1, \omega_2, \omega_3) \sin(c ||\vec{\omega}|| t),$$

where $||\vec{\omega}|| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$. To compute the inverse Fourier transform, we first compute

$$\mathcal{F}^{-1} \left[ \frac{\sin(c ||\vec{\omega}|| t)}{c ||\vec{\omega}||} \right] = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{i r \rho \cos \phi} \sin(cpt) \frac{\sin(cpt)}{c\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{c \sqrt{2\pi}} \int_0^{\infty} \rho \sin(cpt) \int_0^{\pi} e^{i r \rho \cos \phi} \sin \phi \, d\phi \, d\rho$$

$$= \frac{1}{c \sqrt{2\pi}} \int_0^{\infty} \rho \sin(cpt) \int_{-1}^{1} e^{i r u \cos \phi} \, du \, d\rho, \quad u = \cos \phi$$

$$= \frac{1}{c \sqrt{2\pi}} \int_0^{\infty} \sin(cpt) \frac{1}{i r} [e^{ir} - e^{-ir}] \, d\rho$$

$$= \frac{1}{c r \sqrt{2\pi}} \int_0^{\infty} \sin(cpt) \sin(r \rho) \, d\rho$$

$$= \frac{1}{c r \sqrt{2\pi}} \int_0^{\infty} \cos[r(r - ct)] - \cos[r(r + ct)] \, d\rho$$

$$= \frac{\pi}{c r \sqrt{2\pi}} \delta(r - ct)$$
where \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \delta(x) \) is the Dirac delta function. We have used spherical coordinates, with the “north pole” \( \phi = 0 \) pointing in the direction of the position vector \((x, y, z) = (0, 0, r)\). We then apply the convolution theorem to obtain

\[
u(x, y, z, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi(x', y', z') \left[ \frac{\pi}{cr\sqrt{2\pi}} \delta(r' - ct) \right] dx' dy' dz' = \frac{1}{4\pi c^2 t^2} \int_0^{2\pi} \int_0^\pi \psi(x + ct \cos \theta \sin \phi, y + ct \sin \theta \sin \phi, z + ct \cos \phi)(ct)^2 \sin \phi d\phi d\theta - r',
\]

where \( r' = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \) and

\[
\overline{\psi} = \frac{1}{4\pi c^2 t^2} \int_0^{2\pi} \int_0^\pi \psi(x + ct \cos \theta \sin \phi, y + ct \sin \theta \sin \phi, z + ct \cos \phi)(ct)^2 \sin \phi d\phi d\theta
\]
is the average of \( \psi \) over the sphere of center \((x, y, z)\) and radius \( ct \).

Now, we consider the IVP

\[
\begin{align*}
\partial_{tt} u & = c^2 \nabla^2 u, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \\
u(x, y, z, 0) & = \phi(x, y, z), \quad \partial_t u(x, y, z, 0) = 0.
\end{align*}
\]

Let \( v \) be the solution of

\[
\begin{align*}
\partial_{tt} v & = c^2 \nabla^2 v, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \\
v(x, y, z, 0) & = 0, \quad \partial_t v(x, y, z, 0) = \phi(x, y, z).
\end{align*}
\]

Then

\[
v(x, y, z, t) = t\overline{\phi},
\]

where \( \overline{\phi} \) is the average of \( \phi \) over the sphere centered at \((x, y, z)\) with radius \( ct \). We then have

\[
(v_t)_{tt} = (v_{tt})_t = (c^2 \nabla^2 v)_t = c^2 \nabla^2 (v_t),
\]

and

\[
v_t(x, y, z, 0) = \phi(x, y, z), \quad (v_t)_t(x, y, z, 0) = 2\overline{\phi}_t|_{t=0} = 0,
\]

which means that \( u = v_t \), as it satisfies the PDE and initial conditions of the IVP for \( u \).

We conclude that the solution of the IVP

\[
\begin{align*}
\partial_{tt} u & = c^2 \nabla^2 u, \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0, \\
u(x, y, z, 0) & = \phi(x, y, z), \quad \partial_t u(x, y, z, 0) = \psi,
\end{align*}
\]
is

\[
\partial_{tt} u = \frac{\partial}{\partial t} [t\overline{\phi}] + t\overline{\psi}.
\]

This formula for the solution is known as Kirchhoff’s formula, as well as Poisson’s formula.

Because the solution depends on integrals over a sphere of radius \( ct \), it follows that if the initial data are zero except within a small sphere, then the solution is zero at any point \((x_0, y_0, z_0)\) outside this sphere until \( ct \) is large enough so that the sphere centered at \((x_0, y_0, z_0)\) with radius \( ct \) overlaps the sphere within which the initial data is nonzero. That is, the solution exhibits a sharp leading edge. Then, once \( ct \) is so large that the sphere centered at \((x_0, y_0, z_0)\) with radius \( ct \) contains the sphere within which the initial data is nonzero, the solution at \((x_0, y_0, z_0)\) is zero again, and will remain zero. That is, the solution also exhibits a sharp trailing edge. This is known as Huygen’s principle. It holds in dimensions 3, 5, 7, and so on, but not in another dimension. We have already seen that it does not hold in one dimension, as D’Alembert’s solution integrates \( u_t(x, 0) = g(x) \) from \( x - ct \) to \( x + ct \).
Two-Dimensional Wave Equation

The solution of the wave equation in two dimensions can be obtained by solving the three-dimensional wave equation in the case where the initial data depends only on \(x\) and \(y\), but not \(z\). In this case, the three-dimensional solution consists of cylindrical waves.

We first consider the IVP
\[
\begin{align*}
   u_{tt} &= c^2 \nabla^2 u, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\
   u(x, y, 0) &= 0, \quad u_t(x, y) = \psi(x, y).
\end{align*}
\]
Extending this problem to three dimensions, we obtain the solution
\[
   u = t \psi,
\]
where
\[
   \psi = \frac{1}{4\pi c^2 t^2} \int_0^{2\pi} \int_0^{\pi} \psi(x + ct \cos \theta \sin \phi, y + ct \sin \theta \sin \phi, z + ct \cos \phi)(ct)^2 \sin \phi \, d\phi \, d\theta.
\]
To obtain the solution of the original two-dimensional problem, we need to convert this integral over a sphere of radius \(ct\) to an integral over a disc of radius \(ct\). This approach to obtaining the solution is called the method of descent.

We use a substitution to change the spherical coordinate variable \(\phi\) to the polar coordinate variable \(r\). From the relation between them,
\[
   \phi = \cos^{-1} \left( \frac{z}{ct} \right) = \cos^{-1} \left( \frac{\sqrt{(ct)^2 - r^2}}{ct} \right)
\]
we obtain
\[
   d\phi = \frac{1}{\sqrt{1 - \frac{(ct)^2 - r^2}{(ct)^2}}} \frac{1}{ct} \frac{1}{2} \frac{1}{\sqrt{(ct)^2 - r^2}} (-2r) = \frac{1}{\sqrt{(ct)^2 - r^2}} \, dr.
\]
From
\[
   \sin \phi = \frac{r}{ct}
\]
our solution becomes
\[
   t \psi = \frac{t}{4\pi c^2 t^2} \int_0^{2\pi} \int_0^{\pi} \psi(x + ct \cos \theta \sin \phi, y + ct \sin \theta \sin \phi)(ct)^2 \sin \phi \, d\phi \, d\theta
\]
\[
   = \frac{2}{4\pi c^2 t} \int_0^{2\pi} \int_0^{ct} \frac{\psi(x + r \cos \theta, y + r \sin \theta)}{\sqrt{(ct)^2 - r^2}} \frac{r}{ct} \, dr \, d\theta
\]
\[
   = \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{\psi(x + r \cos \theta, y + r \sin \theta)}{\sqrt{(ct)^2 - r^2}} r \, dr \, d\theta.
\]
It follows from Kirchhoff’s formula that the solution of the IVP
\[
\begin{align*}
   u_{tt} &= c^2 \nabla^2 u, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\
   u(x, y, 0) &= \phi(x, y), \quad u_t(x, y) = \psi(x, y).
\end{align*}
\]
is
\[
   u(x, t) = \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{\psi(x + r \cos \theta, y + r \sin \theta)}{\sqrt{(ct)^2 - r^2}} r \, dr \, d\theta + \\
   \frac{\partial}{\partial t} \left[ \frac{1}{2\pi c} \int_0^{2\pi} \int_0^{ct} \frac{\phi(x + r \cos \theta, y + r \sin \theta)}{\sqrt{(ct)^2 - r^2}} r \, dr \, d\theta \right].
\]
Because these integrals are over the interior of the disc, as opposed to the boundary of a sphere in the three-dimensional case, it can be concluded that Huygen’s principle does not hold in two dimensions.