These notes correspond to Lesson 9 in the text.

Solving Nonhomogeneous PDEs

Separation of variables can only be applied directly to homogeneous PDE. However, it can be generalized to nonhomogeneous PDE with homogeneous boundary conditions by solving nonhomogeneous ODE in time.

We consider a general diffusive, second-order, self-adjoint linear IBVP of the form

\[ u_t = (p(x)u_x)_x - q(x)u + f(x,t), \quad 0 < x < L, \]

\[ \alpha_1 u(0,t) + \beta_1 u_x(0,t) = 0, \quad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0, \quad t > 0, \]

\[ u(x,0) = \phi(x), \quad 0 < x < L. \]

We assume that the corresponding homogeneous PDE,

\[ v_t = (p(x)v'_x)_x - q(x)v, \quad 0 < x < L, \]

has a solution of the form

\[ v(x,t) = X(x)S(t). \]

Substituting into the PDE yields

\[ S'(t)X(x) = S(t)(p(x)X'(x))' - q(x)S(t)X(x). \]

Rearranging yields

\[ \frac{S'(t)}{S(t)} = \frac{(p(x)X'(x))' - q(x)X(x)}{X(x)} = -\lambda, \]

where \( \lambda \) is a constant. We then obtain the ODE

\[ S'(t) + \lambda S(t) = 0, \]

\[ -(p(x)X'(x))' + q(x)X(x) = \lambda X(x). \]

The ODE for \( X(x) \) is a Sturm-Liouville problem, which has eigenvalues \( \{\lambda_n\}_{n=1}^{\infty} \) and corresponding eigenfunctions \( \{X_n(x)\}_{n=1}^{\infty} \) that satisfy the boundary conditions. Furthermore, these eigenfunctions are orthogonal, in the sense that

\[ \int_0^L X_n(x)X_m(x) \, dx = 0, \quad m \neq n. \]

They also form a basis for the function space consisting of functions that satisfy the boundary conditions, meaning that any such function can be expressed as a linear combination of these eigenfunctions.
Returning to the original nonhomogeneous PDE, we expand the solution $u(x, t)$ and the external source term $f(x, t)$ in the basis of eigenfunctions. That is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t)X_n(x), \quad f(x, t) = \sum_{n=1}^{\infty} f_n(t)X_n(x),$$

where

$$u_n(t) = \frac{\int_0^L u(x, t)X_n(x) \, dx}{\int_0^L X_n(x)^2 \, dx}, \quad f_n(t) = \frac{\int_0^L f(x, t)X_n(x) \, dx}{\int_0^L X_n(x)^2 \, dx}.$$

Then, to solve the PDE, we multiply both sides by $X_m(x)$ and integrate from 0 to $L$. This yields

$$\int_0^L u_t(x, t)X_m(x) \, dx = \int_0^L (p(x)u_x(x, t))_x - q(x)u(x, t)X_m(x) \, dx + \int_0^L f(x, t)X_m(x) \, dx.$$

Applying the above expansions, and the fact that $X_m(x)$ is an eigenfunction of the spatial ODE, yields

$$u'_m(t) = -\lambda_m u_m(t) + f_m(t),$$

and from the initial condition, we obtain

$$u_m(0) = a_m = \frac{\int_0^L \phi(x)X_m(x) \, dx}{\int_0^L X_m(x)^2 \, dx}.$$

This nonhomogeneous ODE can be solved using an integrating factor $\mu(t) = e^{\lambda_m t}$. We then have

$$u_m(t) = e^{-\lambda_m t}a_m + \int_0^t e^{-\lambda_m(t-s)}f_m(s) \, ds.$$

In summary, the solution of the IBVP is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t)X_n(x),$$

where $X_n(x)$ is the solution of

$$-(p(x)X'_n(x))' + q(x)X_n(x) = \lambda_n X_n(x)$$

that also satisfies the boundary conditions, and the coefficients $u_n(t)$ of the eigenfunction expansion of $u(x, t)$ are given by

$$u_n(t) = e^{-\lambda_n t}a_n + \int_0^t e^{-\lambda_n(t-s)}f_n(s) \, ds,$$

where

$$a_n = \frac{\int_0^L \phi(x)X_n(x) \, dx}{\int_0^L X_n(x)^2 \, dx}, \quad f_n(t) = \frac{\int_0^L f(x, t)X_n(x) \, dx}{\int_0^L X_n(x)^2 \, dx}.$$

This approach to finding $u(x, t)$ is known as the method of eigenfunction expansions.

**Example** We will use the method of eigenfunction expansions to solve the IBVP

$$u_t = \alpha^2 u_{xx} + \sin(3\pi x), \quad 0 < x < 1,$$
\begin{align*}
u(0, t) = 0, & \quad u(1, t) = 0, \quad t > 0, \\
u(x, 0) = \sin(\pi x), & \quad 0 < x < 1.
\end{align*}

Applying separation of variables to the corresponding homogeneous PDE,

\[ u_t = \alpha^2 u_{xx}, \]

yields the ODE

\[ T'' + \alpha^2 \lambda T = 0, \]
\[ X'' + \lambda X = 0, \]

with boundary conditions

\[ X(0) = 0, \quad X(1) = 0. \]

The spatial ODE has the solutions, or eigenfunctions,

\[ X_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots, \]

with corresponding eigenvalues \( \lambda_n = (n\pi)^2, \quad n = 1, 2, \ldots. \)

Expanding the source term, initial data, and solution in the basis of eigenfunctions yields

\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x), \]
\[ \sin(\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \]
\[ \sin(3\pi x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x), \]

where, because of the orthogonality of the eigenfunctions, \( a_1 = 1, \quad f_3 = 1, \quad \text{and} \quad a_n = 0 \quad \text{for all} \quad n \neq 1 \)

and \( f_n = 0 \quad \text{for all} \quad n \neq 3. \)

From the formula

\[ u_n(t) = e^{-(n\pi\alpha)^2 t} a_n + \int_0^t e^{-(n\pi\alpha)^2 (t-s)} f_n \, ds, \]

we obtain

\[ u_1(t) = e^{-(\pi\alpha)^2 t} a_1 = e^{-(\pi\alpha)^2 t}, \]
\[ u_3(t) = e^{-(3\pi\alpha)^2 t} a_3 + f_3 \int_0^t e^{-(3\pi\alpha)^2 (t-s)} \, ds \]
\[ = \frac{1}{(3\pi\alpha)^2} e^{-(3\pi\alpha)^2 t} e^{(3\pi\alpha)^2 t} \Bigg|_0^t \]
\[ = \frac{1}{(3\pi\alpha)^2} [1 - e^{-(3\pi\alpha)^2 t}], \]

and \( u_n(t) = 0 \) for all \( n \neq 1, 3. \) We conclude that the solution of the IBVP is

\[ u(x, t) = e^{-(\pi\alpha)^2 t} \sin \pi x + \frac{1}{(3\pi\alpha)^2} [1 - e^{-(3\pi\alpha)^2 t}] \sin 3\pi x. \]

Letting \( t \to \infty, \) we obtain the steady-state solution \( \frac{1}{(3\pi\alpha)^2} \sin 3\pi x. \) \( \square \)