These notes correspond to Section 1.1 in the text.

Introduction

This course is about optimization, which is the computation of the minimum or maximum values of an objective function \( f(x) \) of one or several variables, which may or may not be subject to constraints. Optimization has many applications, such as in business, where quantities such as revenue or profit need to be maximized while other quantities such as costs need to be minimized.

Example Consider the function \( f(x, y) = x^2y \). We wish to find the maximum value of this function, subject to the constraints that \( x + y = 50 \) and \( x \geq 0 \). This can be accomplished by solving for \( y \) in terms of \( x \), which yields \( y = 50 - x \), and then maximizing the function \( g(x) = x^2(50-x) = 50x^2 - x^3 \).

From \( g'(x) = 100x - 3x^2 \), \( g''(x) = 100 - 6x \), we obtain the critical points \( x = 0 \) and \( x = 100/3 \). Because \( g''(0) = 100 \) and \( g''(100/3) = -100 \), we conclude that the maximum occurs at either \( x = 100/3 \) or the endpoint \( x = 0 \). Substituting these values into \( g(x) \) yields \( g(0) = 0 \) and \( g(100/3) = 1000/54 \), so the maximum occurs at \( x = 100/3 \). □

Functions of One Variable

We can obtain conditions that \( f(x) \) and its derivatives must satisfy at a minimum or maximum using Taylor’s Theorem.

Theorem (Taylor’s Formula) Suppose that \( f(x) \), \( f'(x) \) and \( f''(x) \) exist on the closed interval \([a,b]=\{x \in \mathbb{R} \mid a \leq x \leq b \}\). If \( x^*, x \in [a,b], x \neq x^* \), then there exists a point \( z \) between \( x \) and \( x^* \) such that

\[
  f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} f''(z)(x - x^*)^2.
\]

It follows from this formula that if \( f'(x^*) = 0 \) and \( f''(x) > 0 \), then \( f(x) = f(x^*) + d \) where \( d \) is a positive number. That is, \( f(x) > f(x^*) \). Therefore, if \( f''(x) > 0 \) for all \( x \) in the domain of \( f \), then \( x^* \) is the point at which the value of \( f(x) \) is minimized. Similarly, if \( f''(x) < 0 \) for all \( x \) in the domain of \( f \), then \( x^* \) is the point at which the value of \( f(x) \) is maximized. This observation is known as the Second Derivative Test.
Example Let $f(x) = e^{x^2}$. Then we have

$$f'(x) = 2xe^{x^2}, \quad f''(x) = 2(2x^1 + 1)e^{x^2}.$$ 

The only point $x^*$ at which $f'(x^*) = 0$ is at $x^* = 0$. Because $f''(x) > 0$, it follows from the preceding theorem that $f(x) > f(0)$ for all $x \neq 0$. □

We now classify types of points at which the value of a function is minimized or maximized.

**Definition** Suppose $f(x)$ is a real-valued function defined on an interval $I$. A point $x^* \in I$ is a

1. global minimizer of $f(x)$ on $I$ if $f(x^*) \leq f(x)$ for all $x \in I$;
2. strict global minimizer of $f(x)$ on $I$ if $f(x^*) < f(x)$ for all $x \in I$;
3. local minimizer of $f(x)$ if there exists $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in I$ whenever $|x - x^*| < \delta$;
4. strict local minimizer of $f(x)$ if there exists $\delta > 0$ such that $f(x^*) < f(x)$ for all $x \in I$ whenever $0 < |x - x^*| < \delta$;
5. critical point of $f(x)$ if $f'(x^*)$ exists and $f'(x^*) = 0$.

Similar types of maximizers of $f(x)$ can be defined in an entirely analogous manner.

The following theorem helps us to characterize minimizers and maximizers. **Theorem (Fermat)** Suppose that $f(x)$ is differentiable on an interval $I$. If $x^*$ is a local minimizer or maximizer of $f(x)$, then either $x^*$ is an endpoint of $I$, or a critical point of $f(x)$.

This theorem can be proved using the definition of the derivative as a limit.

The following theorem is useful for classifying critical points as minimizers or maximizers.

**Theorem** Suppose that $f(x)$, $f'(x)$ and $f''(x)$ are all continuous on an interval $I$ and that $x^* \in I$ is a critical point of $f(x)$.

1. If $f''(x) \geq 0$ for all $x \in I$, then $x^*$ is a global minimizer of $f(x)$ on $I$.
2. If $f''(x) > 0$ for all $x \in I$ such that $x \neq x^*$, then $x^*$ is a strict global minimizer of $f(x)$ on $I$.
3. If $f''(x^*) > 0$, then $x^*$ is a strict local minimizer of $f(x)$.

This theorem can be proved using Taylor’s Formula. The third part uses the continuity of $f''(x)$ in conjunction with Taylor’s Formula.
**Example** Let \( f(x) = 3x^4 - 4x^3 + 1 \). From
\[
f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1), \quad f''(x) = 36x^2 - 24x = 12x(3x - 2),
\]
we obtain the critical points \( x = 0, 1 \). From \( f''(0) = 0 \) and \( f''(1) = 12 \), we see that \( f \) has a strict local minimizer at \( x = 1 \). Because \( f''(x) \) changes sign near \( x = 0 \), this point is an *inflection point* of \( f(x) \). However, \( f'(x) \) does not change sign near \( x = 0 \), so this point is neither a minimizer nor a maximizer. Finally, because
\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty,
\]
we can conclude that \( x = 1 \) is also a strict *global* minimizer of \( f(x) \). \qed

**Example** Let \( f(x) = \ln(1 - x^2) \) on \((-1, 1)\). From
\[
f'(x) = \frac{-2x}{1 - x^2}, \quad f''(x) = \frac{-2 - 2x^2}{1 - x^2},
\]
we obtain the critical point \( x = 0 \). Because \( f''(x) < 0 \) on the entire interval \((-1, 1)\), it follows from the preceding theorem that \( x = 0 \) is a strict global maximizer of \( f(x) \). \qed

**Exercises**

1. Chapter 1, Exercise 1