These notes correspond to Section 3.2 in the text.

**Interpolation Using Equally Spaced Points**

Suppose that the interpolation points \(x_0, x_1, \ldots, x_n\) are equally spaced; that is, \(x_i = x_0 + ih\) for some positive number \(h\). In this case, the Newton interpolating polynomial can be simplified, since the denominators of all of the divided differences can be expressed in terms of the spacing \(h\). If we recall the *forward difference operator* \(\Delta\), defined by

\[
\Delta x_k = x_{k+1} - x_k,
\]

where \(\{x_k\}\) is any sequence, then the divided differences \(f[x_0, x_1, \ldots, x_k]\) are given by

\[
f[x_0, x_1, \ldots, x_k] = \frac{1}{k! h^k} \Delta^k f(x_0).
\]

The interpolating polynomial can then be described by the *Newton forward-difference formula*

\[
p_n(x) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0),
\]

where the new variable \(s\) is related to \(x\) by

\[
s = \frac{x - x_0}{h},
\]

and the *extended binomial coefficient* \(\binom{s}{k}\) is defined by

\[
\binom{s}{k} = \frac{s(s-1)(s-2)\cdots(s-k+1)}{k!},
\]

where \(k\) is a nonnegative integer.

**Example** We will use the *Newton forward-difference formula*

\[
p_n(x) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)
\]

to compute the interpolating polynomial \(p_3(x)\) that fits the data
In other words, we must have \( p_3(-1) = 3 \), \( p_3(0) = -4 \), \( p_3(1) = 5 \), and \( p_3(2) = -6 \). Note that the interpolation points \( x_0 = -1 \), \( x_1 = 0 \), \( x_2 = 1 \) and \( x_3 = 2 \) are equally spaced, with spacing \( h = 1 \).

To apply the forward-difference formula, we define \( s = (x - x_0)/h = x + 1 \) and compute

\[
\begin{array}{ll}
\begin{array}{l}
\left( \begin{array}{l}
\ s \\
\ 1 \\
\end{array} \right) \\
= s \\
= x + 1, \\
\end{array} \\
\left( \begin{array}{l}
\ s \\
\ 2 \\
\end{array} \right) \\
= \frac{s(s-1)}{2} \\
= \frac{x(x+1)}{2}, \\
\left( \begin{array}{l}
\ s \\
\ 3 \\
\end{array} \right) \\
= \frac{s(s-1)(s-2)}{6} \\
= \frac{(x+1)x(x-1)}{6}, \\
\end{array}
\]

\[
\begin{array}{l}
f[x_0] = f(x_0) \\
= 3, \\
\Delta f(x_0) = f(x_1) - f(x_0) \\
= -4 - 3 \\
= -7, \\
\Delta^2 f(x_0) = \Delta(\Delta f(x_0)) \\
= \Delta[f(x_1) - f(x_0)] \\
= [f(x_2) - f(x_1)] - [f(x_1) - f(x_0)] \\
= f(x_2) - 2f(x_1) + f(x_0) \\
= 5 - 2(-4) + 3, \\
= 16 \\
\Delta^3 f(x_0) = \Delta(\Delta^2 f(x_0)) \\
= \Delta[f(x_2) - 2f(x_1) + f(x_0)] \\
= [f(x_3) - f(x_2)] - 2[f(x_2) - f(x_1)] + [f(x_1) - f(x_0)] \\
= f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0) \\
= -6 - 3(5) + 3(-4) - 3
\end{array}
\]
\[ p_3(x) = f[x_0] + \sum_{k=1}^{3} \binom{s}{k} \Delta^k f(x_0) \]

\[ = 3 + \binom{s}{1} \Delta f(x_0) + \binom{s}{2} \Delta^2 f(x_0) + \binom{s}{3} \Delta^3 f(x_0) \]

\[ = 3 + (x + 1)(-7) + \frac{x(x + 1)}{2} 16 + \frac{(x + 1)x(x - 1)}{6} (-36) \]

\[ = 3 - 7(x + 1) + 8(x + 1)x - 6(x + 1)x(x - 1). \]

Note that the forward-difference formula computes the same form of the interpolating polynomial as the Newton divided-difference formula. □

If we define the backward difference operator \( \nabla \) by

\[ \nabla x_k = x_k - x_{k-1}, \]

for any sequence \( \{x_k\} \), then we obtain the Newton backward-difference formula

\[ p_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n), \]

where \( s = (x - x_n)/h \), and the preceding definition of the extended binomial coefficient applies.

**Example** We will use the Newton backward-difference formula

\[ p_n(x) = f[x_n] + \sum_{k=1}^{n} (-1)^k \binom{-s}{k} \nabla^k f(x_n) \]

to compute the interpolating polynomial \( p_3(x) \) that fits the data

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f(x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>-6</td>
</tr>
</tbody>
</table>

In other words, we must have \( p_3(-1) = 3, p_3(0) = -4, p_3(1) = 5, \) and \( p_3(2) = -6. \) Note that the interpolation points \( x_0 = -1, x_1 = 0, x_2 = 1 \) and \( x_3 = 2 \) are equally spaced, with spacing \( h = 1. \)
To apply the backward-difference formula, we define $s = (x - x_3)/h = x - 2$ and compute

\[
\begin{pmatrix}
-s \\
1
\end{pmatrix} = -s = -(x - 2),
\begin{pmatrix}
-s \\
2
\end{pmatrix} = -s(-s - 1) = \frac{s(s + 1)}{2} = \frac{(x - 2)(x - 1)}{2},
\begin{pmatrix}
-s \\
3
\end{pmatrix} = -s(-s - 1)(-s - 2) = \frac{s(s + 1)(s + 2)}{6} = \frac{(x - 2)(x - 1)x}{6},
\]

\[
f[x_3] = f(x_3)
= -6,
\nabla f(x_3) = f(x_3) - f(x_2)
= -6 - 5
= -11,
\nabla^2 f(x_3) = \nabla(\nabla f(x_3))
= \Delta[f(x_3) - f(x_2)]
= [f(x_3) - f(x_2)] - [f(x - 2) - f(x_1)]
= f(x_3) - 2f(x_2) + f(x_1)
= -6 - 2(5) + 4
= -20
\]

\[
\nabla^3 f(x_3) = \nabla(\nabla^2 f(x_3))
= \nabla[f(x_3) - 2f(x_2) + f(x_1)]
= [f(x_3) - f(x_2)] - 2[f(x_2) - f(x_1)] + [f(x_1) - f(x_0)]
= f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)
= -6 - 3(5) + 3(-4) - 3
= -36.
\]
It follows that
\[
p_3(x) = f[x_3] + \sum_{k=1}^{3} (-1)^k \binom{-s}{k} \nabla^k f(x_3) \\
= -6 - \binom{-s}{1} \nabla f(x_3) + \binom{-s}{2} \nabla^2 f(x_3) - \binom{-s}{3} \nabla^3 f(x_3) \\
= -6 - [-(x - 2)](-11) + \frac{(x - 2)(x - 1)}{2}(-20) - \frac{(x - 2)(x - 1)x}{6}(-36) \\
= -6 - 11(x - 2) - 10(x - 2)(x - 1) - 6(x - 2)(x - 1)x.
\]

Note that the backward-difference formula does not compute the same form of the interpolating polynomial \(p_n(x)\) as the Newton divided-difference formula. Instead, it computes a different Newton form of \(p_n(x)\) given by
\[
p_n(x) = \sum_{j=0}^{n} f[x_j, x_{j+1}, \ldots, x_n] \prod_{i=j+1}^{n} (x - x_i) \\
= f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_{n-1})(x - x_n) + \\
\cdots + f[x_0, x_1, \ldots, x_n](x - x_1)(x - x_2) \cdots (x - x_{n-1})(x - x_n).
\]

The divided-differences \(f[x_j, \ldots, x_n]\), for \(j = 0, 1, \ldots, n\), can be obtained by constructing a divided-difference table as in the first example. These divided differences appear in the bottom row of the table, whereas the divided differences used in the forward-difference formula appear in the top row. □