Richardson Extrapolation

We have seen that the accuracy of methods for computing derivatives of a function $f(x)$ depends on the spacing between points at which $f$ is evaluated, and that the approximation tends to the exact value as this spacing tends to 0.

Suppose that a uniform spacing $h$ is used. We denote by $F(h)$ the approximation computed using the spacing $h$, from which it follows that the exact value is given by $F(0)$. Let $p$ be the order of accuracy in our approximation; that is,

$$F(h) = a_0 + a_1 h^p + O(h^r), \quad r > p,$$

where $a_0$ is the exact value $F(0)$. Then, if we choose a value for $h$ and compute $F(h)$ and $F(h/q)$ for some positive integer $q$, then we can neglect the $O(h^r)$ terms and solve a system of two equations for the unknowns $a_0$ and $a_1$, thus obtaining an approximation that is $r$th order accurate. If we can describe the error in this approximation in the same way that we can describe the error in our original approximation $F(h)$, we can repeat this process to obtain an approximation that is even more accurate.

This process of extrapolating from $F(h)$ and $F(h/q)$ to approximate $F(0)$ with a higher order of accuracy is called Richardson extrapolation. In a sense, Richardson extrapolation is similar in spirit to Aitken’s $\Delta^2$ method, as both methods use assumptions about the convergence of a sequence of approximations to “solve” for the exact solution, resulting in a more accurate method of computing approximations.

**Example** Consider the function

$$f(x) = \frac{\sin^2 \left( \frac{\sqrt{x^2 + 2}}{\cos x^2 - 2} \right)}{\sin \left( \frac{\sqrt{x^2 - 1}}{\sqrt{x^2 + 1}} \right)}.$$

Our goal is to compute $f'(0.25)$ as accurately as possible. Using a centered difference approximation,

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2),$$

with $x = 0.25$ and $h = 0.01$, we obtain the approximation

$$f'(0.25) \approx \frac{f(0.26) - f(0.24)}{0.02} = -9.06975297890147,$$
which has absolute error $3.0 \times 10^{-3}$, and if we use $h = 0.005$, we obtain the approximation

$$f'(0.25) \approx \frac{f(0.255) - f(0.245)}{0.01} = -9.06746429492149,$$

which has absolute error $7.7 \times 10^{-4}$. As expected, the error decreases by a factor of approximately 4 when we halve the step size $h$, because the error in the centered difference formula is of $O(h^2)$.

We can obtain a more accurate approximation by applying Richardson Extrapolation to these approximations. We define the function $N_1(h)$ to be the centered difference approximation to $f'(0.25)$ obtained using the step size $h$. Then, with $h = 0.01$, we have

$$N_1(h) = -9.06975297890147, \quad N_1(h/2) = -9.066746429492149,$$

and the exact value is given by $N_1(0) = -9.06669877124279$. Because the error in the centered difference approximation satisfies

$$N_1(h) = N_1(0) + K_1 h^2 + K_2 h^4 + K_3 h^6 + O(h^8),$$

where the constants $K_1$, $K_2$ and $K_3$ depend on the derivatives of $f(x)$ at $x = 0.25$, it follows that the new approximation

$$N_2(h) = N_1(h/2) + \frac{N_1(h/2) - N_1(h)}{2^2 - 1} = -9.06670140026149,$$

has fourth-order accuracy. Specifically, if we denote the exact value by $N_2(0)$, we have

$$N_2(h) = N_2(0) + \tilde{K}_2 h^4 + \tilde{K}_3 h^6 + O(h^8),$$

where the constants $\tilde{K}_2$ and $\tilde{K}_3$ are independent of $h$.

Now, suppose that we compute

$$N_1(h/4) = \frac{f(x + h/4) - f(x - h/4)}{2(h/4)} = \frac{f(0.2525) - f(0.2475)}{0.005} = -9.06689027527046,$$

which has an absolute error of $1.9 \times 10^{-4}$, we can use extrapolation again to obtain a second fourth-order accurate approximation,

$$N_2(h/2) = N_1(h/4) + \frac{N_1(h/4) - N_1(h/2)}{3} = -9.06669893538678,$$

which has absolute error of $1.7 \times 10^{-7}$. It follows from the form of the error in $N_2(h)$ that we can use extrapolation on $N_2(h)$ and $N_2(h/2)$ to obtain a sixth-order accurate approximation,

$$N_3(h) = N_2(h/2) + \frac{N_2(h/2) - N_2(h)}{2^4 - 1} = -9.06669877106180,$$

which has an absolute error of $1.8 \times 10^{-10}$. □