This assignment is due in class on Tuesday, September 1.

1. Use the Intermediate Value Theorem and Rolle’s Theorem to show that the graph of \( f(x) = x^3 + 2x + k \) crosses the x-axis exactly once, regardless of the value of the constant \( k \). \textit{Hint:} Use the derivative.

2. Find the second Taylor polynomial \( P_2(x) \) for the function \( f(x) = e^x \cos x \) about \( x_0 = 0 \).
   
   (a) Use \( P_2(0.5) \) to approximate \( f(0.5) \). Find an upper bound on the error \( |f(0.5) - P_2(0.5)| \) using Taylor’s Theorem, and compare it to the actual error.
   
   (b) Find an upper bound for the error \( |f(x) - P_2(x)| \) in using \( P_2(x) \) to approximate \( f(x) \) on the interval \([0, 1]\).

3. Write an algorithm that accepts a real number \( x \) and a nonnegative integer \( n \) and computes an approximation to \( e^x \) using the \( n \)th Taylor polynomial \( P_n(x) \) with center \( x_0 = 0 \). Find a value of \( n \) necessary for \( P_n(x) \) to approximate \( e^x \) to within \( 10^{-6} \) on \([0, 0.5]\). Show that with this value of \( n \), your algorithm approximates \( e^x \) to this degree of accuracy at \( x = 0.25 \).

4. A function \( f : [a, b] \to \mathbb{R} \) is said to satisfy a \textit{Lipschitz condition} with Lipschitz constant \( L \) on \([a, b]\) if, for every \( x, y \in [a, b] \), we have \( |f(x) - f(y)| \leq L|x - y| \). Show that if \( f \) satisfies a Lipschitz condition with Lipschitz constant \( L \) on an interval \([a, b]\), then \( f \) is continuous on \([a, b]\).