This exam is due by 5pm on Tuesday, April 15. You may use any resource, including the textbook, course notes, or other published information, but all sources outside of course material must be cited. You may not collaborate with anyone.

1. Write a MATLAB function that solves tridiagonal systems of equations of size \( n \). Assume that no pivoting is needed, but do not assume that the tridiagonal matrix \( A \) is symmetric. Your program should expect as input four vectors of size \( n \) (or \( n - 1 \)): one right hand side \( b \) and the three nonzero diagonals of \( A \). It should calculate and return \( x = A^{-1}b \) using a Gaussian elimination variant which requires \( O(n) \) flops and consumes no additional space as a function of \( n \) (i.e. in total \( 5n \) storage locations are required).

Apply your program for the problem \( A v = g \), where

\[
\begin{bmatrix}
v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ v_N
\end{bmatrix}, \quad \begin{bmatrix}
g(t_1) \\ g(t_2) \\ \vdots \\ g(t_{N-1}) \\ g(t_N)
\end{bmatrix}, \quad A = \frac{1}{h^2} \begin{bmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -2 & 2
\end{bmatrix},
\]

where \( N = 100 \), \( t_i = ih \) with \( h = 1/N \), and \( g(t) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}t\right) \). Compare the results to the vector \( u \) composed of \( u(ih) = \sin\left(\frac{\pi}{2}ih\right), \quad i = 1, \ldots, N \), by recording \( \max |v - u| \).

2. The \( n \times n \) matrix \( A \) is said to be in Hessenberg form if all its elements below the first subdiagonal are zero. That is, \( a_{ij} = 0, \quad i > j + 1 \).

Consider the LU decomposition of such a matrix, assuming that no pivoting is needed: \( A = LU \).

(a) Provide an efficient algorithm for this LU decomposition.

(b) What is the sparsity structure of the resulting matrix \( L \) (i.e. where are its nonzeros)?

(c) How many operations (to a leading order) does the algorithm require?

3. Given that \( a \) and \( b \) are two real positive numbers, the eigenvalues of the symmetric tridiagonal matrix

\[
A = \begin{bmatrix}
a & b & & \\
b & a & b & \\
& \ddots & \ddots & \ddots \\
& & b & a & b \\
& & & b & a
\end{bmatrix}
\]

of size \( n \times n \) are \( \lambda_j = a + 2b \cos\left(\frac{\pi j}{n+1}\right), \quad j = 1, \ldots, n \). This matrix can be obtained in MATLAB with the instruction

\[
A = \text{diag}(a \times \text{ones}(n,1),0) + \text{diag}(b \times \text{ones}(n-1,1),1) + \text{diag}(b \times \text{ones}(n-1,1),-1).
\]
(a) Find $\|A\|_\infty$.

(b) Show that if $A$ is strictly diagonally dominant then it is symmetric positive definite.

(c) Suppose $a > 0$ and $b > 0$ are such that $A$ is symmetric positive definite. Find the condition number $\kappa_2(A)$. (Assuming that $n$ is large, an approximate value would suffice.)

4. Consider the $2 \times 2$ matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

and suppose we are required to solve $Ax = b$.

(a) Write down explicitly the iteration matrices corresponding to the Jacobi, Gauss-Seidel and SOR schemes.

(b) Find the spectral radius of the Jacobi and Gauss-Seidel iteration matrices.

(c) Plot a graph of the spectral radius of the SOR iteration matrix vs. the relaxation parameter $\omega$, for $0 \leq \omega \leq 2$.

(d) Find the optimal SOR parameter, $\omega^*$. What is the spectral radius of the corresponding iteration matrix?

5. The linear system

$$
\begin{align*}
10x_1 + x_2 + x_3 &= 12 \\
x_1 + 10x_2 + x_3 &= 12 \\
x_1 + x_2 + 10x_3 &= 12
\end{align*}
$$

has the unique solution $x_1 = x_2 = x_3 = 1$.

(a) Starting from $x^{(0)} = (0, 0, 0)^T$ perform
- two iterations using Jacobis method
- two iterations using Gauss-Seidel.

Calculate error norms for the four iterations in $\ell_1$. Which method seems to converge faster?

(b) Show that Jacobi’s method will converge for this iteration regardless of the starting vector $x^{(0)}$.

(c) Now apply two Jacobi iterations for the problem

$$
\begin{align*}
2x_1 + 5x_2 + 5x_3 &= 12 \\
5x_1 + 2x_2 + 5x_3 &= 12 \\
5x_1 + 5x_2 + 2x_3 &= 12
\end{align*}
$$

starting from $x^{(0)} = (0, 0, 0)^T$. Does the method appear to converge? Explain why.

6. Consider the iteration

$$x^{(k+1)} = x^{(k)} + \alpha_k d_k,$$

where $d_k = A^T e_k \neq 0$ is the search direction and $\alpha_k$ is a scalar. Denote the error vectors by $e_k = x - x^{(k)}$, where $x$ is the exact solution.
(a) Determine $\alpha_k$ so that $\|e_{k+1}\|_2$ is minimized. Note: since $e_k$ and $x$ are not known, $\alpha_k$ should not contain those quantities!

(b) Show that for this choice of $\alpha_k$ the error vector $e_{k+1}$ is orthogonal to $d_k$.

(c) Show that $\|e_{k+1}\|_2 \leq \|e_k\|_2$.

(d) Suppose $f_k = r_k$ for all $k$. Determine whether or not the algorithm converges in this case for any arbitrarily given initial guess.

7. (Bonus) For an $m \times n$ matrix $A$ show that

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|.$$