INFINITE EXPONENTIALS

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1. A continued exponential shall be denoted by the symbol

\[ E_{i=0}^n a_i \]

which is an abbreviation for

\[
\begin{array}{c}
  a_n \\
  \vdots \\
  a_2 \\
  a_1 \\
  a_0 \\
\end{array}
\]

that is, each \( a_i \) is used as the exponent of the preceding. This symbol is selected because of its analogy with the symbols for a continued sum and a continued product, namely

\[
\sum_{i=0}^n a_i \quad \text{and} \quad \prod_{i=0}^n a_i.
\]

In the continued exponential, the various \( a_i \) shall be called exponents. A longer symbol which puts the exponents into evidence will sometimes be used when one or more of the exponents are irregular, namely

\[
E(a_0, a_1, a_2, \ldots, a_n) = E_{i=0}^n a_i.
\]

The order of this exponential is \( n \).

2. Some obvious properties of continued exponentials are

(a) \[ E(a_0, a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) = E[a_0, a_1, \ldots, a_m, E(a_{m+1}, \ldots, a_n)] \]

(b) \[ E(a_0, a_1, \ldots, a_m, 1, a_{m+2}, \ldots, a_n) = E(a_0, a_1, \ldots, a_m) \]

(c) \[ E(a_0, a_1, \ldots, a_m, 0, a_{m+2}, \ldots, a_n) = E(a_0, a_1, \ldots, a_{m-1}). \]

3. An infinite exponential is naturally denoted by the symbol

\[ E_{i=0}^\infty a_i \quad \text{or} \quad E(a_0, a_1, \ldots) \]

We shall study only the case where all the \( a_i \) are positive or zero, so that none of the exponentials considered later is negative or complex.

If the limit, as \( n \to \infty \), of \( E_{i=0}^n a_i \) exists and equals say \( k \), we shall say that the infinite exponential \( E_{i=0}^\infty a_i \) converges and has the value \( k \); but if the above limit does not exist, the infinite exponential shall be said to diverge.

The infinite exponential \( E_{i=m}^\infty a_i \) shall be called the \( m \)th residual of \( E_{i=0}^\infty a_i \).

If every residual of an infinite exponential converges, then the infinite exponential shall be called properly convergent; and if an infinite exponential con-
varges but at least one of its residuals diverges, the infinite exponential shall be called *improperly convergent*.

4. Some obvious theorems about infinite exponentials are:

**Theorem 1.** If any one of the residuals converges, the infinite exponential converges.

**Theorem 2.** If any particular residual diverges, then every later residual diverges.

If \( E_{i=0}^n a_i \) grows infinite with \( n \), the corresponding infinite exponential shall be called *properly divergent*; but if \( E_{i=0}^n a_i \) never exceeds a fixed bound as \( n \) grows infinite, and yet fails to approach a limit, then the infinite exponential shall be called *improperly divergent*, or oscillating.

**Theorem 3.** If some exponent in an infinite exponential is less than unity, the exponential must converge or oscillate. For if \( a_m \) is less than unity, the \( m \)th residual would not be greater than unity, and so could not grow infinite.

**Theorem 4.** If all the exponents in an infinite exponential are greater than unity, the value of \( E_{i=0}^n a_i \) is monotone increasing with \( n \).

These two theorems show that the two types of divergence defined above are all inclusive, that is, there is no type of unbounded oscillation.

5. The \( n \)th *margerin function* of \( x \) shall be defined by \( E(a_0, a_1, \ldots, a_n, x) \) and its properties are developed here to be used in proving convergence theorems for infinite exponentials.

6. A graphical process for evaluating the margerin function (or any continued exponential) may be described as follows:

Plot the graphs of the curves

\[
y = a_0^x, \quad y = a_1^x, \quad \ldots, \quad y = a_n^x.
\]

(They are indicated in fig. 1 by the curves \( C_0, C_1, C_2, C_3, C_4 \), and the other curves are to be disregarded for the moment). Draw also the forty-five degree turner line.

Then choose a point \( P \) upon the turner whose abscissa is \( x \), from \( P \) draw a vertical line to meet \( C_n \) in the point \( Q_n \), from \( Q_n \) draw a horizontal line to meet the turner in \( P_n \), from \( P_n \) draw a vertical line to meet \( C_{n-1} \) in \( Q_{n-1} \); from \( Q_{n-1} \) draw a horizontal line to meet the turner in \( P_{n-1} \), and continue until we have used each of the curves in turn beginning with \( C_n \) and ending with \( C_0 \), and the final point \( P_0 \); on the turner, will have an abscissa equal to the desired value of the \( n \)th margerin function.

If we start with an infinitesimal segment \( PP' \) of the turner whose horizontal projection is \( \Delta x \), and carry its points through the above graphical process, we shall end with another infinitesimal segment \( P_0P' \), whose horizontal projection we shall call \( \Delta x_0 \); and moreover at each step of the process, the length of the
segment is multiplied by the slope of the C-curve taken at the corresponding Q-point. Then if we let \( m_i \) denote the slope of \( C_i \) at the point \( Q_i \), we shall have

\[ \Delta x_0 = m_0 \cdot m_1 \cdots m_n \cdot \Delta x \]

and hence the slope of the margerin function is given by

\[ \frac{d}{dx} E(a_0, a_1, \cdots, a_n, x) = m_0 \cdot m_1 \cdots m_n. \]

Graphs of \( y = a^x \) for

\( a = e^x, 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, \) and \( e^{x/2} \)

Also \( x = ty \log_e y \)

The foregoing discussion makes it desirable to mark off the region of the plane in which the curves of the family \( y = a^x \) have slopes numerically greater than unity, and the region in which they have slopes numerically less than unity. We shall call the former region the expanding region, and the latter the contracting region. These two regions are separated by a curve called the neutral locus, at each point of which some member of the family has a slope of plus one or minus one.

To get the neutral locus we set
\[ \frac{dy}{dx} = a^x \log a = \pm 1 \]

from which the equation of the neutral locus is found to be

\[ x = \pm y \log y. \]

Fig. 1 shows part of the family and the neutral locus, and it is easy to see the expanding and contracting regions.

7. The problem of the convergence of infinite exponentials all of whose exponents are equal is now easily solved.* Consider \( E(a, a, \cdots) \). If one approach the question of the convergence of this infinite exponential by evaluating graphically the series

\[ E(a), E(a, a), E(a, a, a), \cdots \]

he will use only the graph of \( y = a^x \) and the turner (see figs. 2, 3, 4, 5), and will be led to announce the following theorem:

**Theorem 5.** (a) If \( e^{1/e} < a \), then \( E(a, a, a, \cdots) \) is properly divergent. (b) If \( e^{-e} \leq a \leq e^{1/e} \), then \( E(a, a, \cdots) \) is convergent and has the value \( k \) which satisfies \( k = a^k \) and \( 1/e \leq k \leq e \). (c) If \( 0 < a < e^{-e} \), then \( E(a, a, \cdots) \) is improperly divergent and as the number of exponents increases in \( E(a, a, \cdots, a) \) this exponential tends to oscillate between two values \( k_1 \) and \( k_2 \) never assuming a value between them, where \( k_1 \) and \( k_2 \) are determined by satisfying

\[ k_1 = a^{k_1} \text{ and } k_2 = a^{k_2} \text{ and } 0 < k_1 < \frac{1}{e} < k_2 < 1. \]

Figures 2, 3, and 4 illustrate parts (a) and (b) of the above theorem, and are self-explanatory. Fig. 5 illustrates part (c). In this figure, the points \( Q_n, Q_{n-2}, \cdots \) approach from the left a limiting position \( Q \), and \( Q_{n-1}, Q_{n-3}, \cdots \) approach from the right a limiting position \( Q' \). The points \( Q \) and \( Q' \) are located as the only points on the graph of \( y = a^x \) which are symmetrical with respect to the turner; and two such points will exist only when \( a < e^{-e} \). Theorem 5 brings into prominence two numbers of fundamental importance in our problem, namely

\[ e^{1/e} = 1.444668 \cdots \]
\[ e^{-e} = .06598803 \cdots \]

which we shall call the upper and lower limits of the interval of convergence. The exponential curve \( y = e^{x/e} \) is tangent to the turner at the point \((e, e)\) and \( y = e^{-ex} \) cuts the turner at right angles at the point \((1/e, 1/e)\). The members of the exponential family in between these two cut the turner at slopes numerically less

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* Since completing the paper, it has come to the attention of the author that Prof. Edmund Landau has treated the case of equal exponents in his university lectures, and obtained the results given in theorem 5; but there appears to be nothing published upon the subject.
than unity, and this is the reason why the exponential in theorem 5 (b) converges.
8. **Lemmas concerning the margerin function.**

**Lemma 1.** The margerin function \( E(a_0, a_1, \ldots, a_n, x) \) is a monotone increasing or decreasing function of \( x \) according as an even or an odd number of the \( a_i \) are less than unity. For

\[
\frac{d}{dx} E(a_0, a_1, \ldots, a_n, x) = E(a_0, a_1, \ldots, a_n, x) \cdot E(a_1, a_2, \ldots, a_n, x) \cdot \ldots \cdot E(a_n, x) \cdot \log a_0 \cdot \log a_1 \cdots \log a_n
\]

and all factors in this product are positive except the logarithms of those \( a_i \) which are less than unity.

**Lemma 2.** If all the \( a_i \) lie in the interval of convergence, that is if

\[
e^{-e} \leq a_i \leq e^{1/e}, \quad i = 0, 1, 2, \ldots
\]

and if \( 0 \leq x \leq e \), then

\[
0 \leq E(a_0, a_1, \ldots, a_n, x) \leq e.
\]

The graphical process, fig. 1, will make this self evident.

**Lemma 3.** If all the \( a_i \) are in the interval of convergence, that is if

\[
e^{-e} \leq a_i \leq e^{1/e}, \quad i = 0, 1, 2, \ldots
\]

then the difference between the largest and smallest values of \( E(a_0, a_1, \ldots, a_n, x) \) in the interval \( 0 \leq x \leq e \) approaches zero as \( n \) grows infinite.

Since the margerin function is monotone increasing or decreasing, we can get the difference between its largest and smallest values by integrating its derivative over the interval. Using the form of this derivative given in Art 6, we are trying to prove

\[
\lim_{n \to \infty} \int_0^x m_0 \cdot m_1 \cdots m_n \cdot dx = 0.
\]

Now referring to fig. 1, and remembering that each \( m_i \) is the slope of a member of the family at the point \( Q_i \), we note the following facts. If a point such as \( Q_1 \) or \( Q_4 \) fall in the contracting region, the corresponding slope \( m_1 \), or \( m_4 \) is numerically less than unity. But if a point like \( Q_2 \) fall in the expanding region, the value of \( m_2 \), is numerically greater than unity. However the preceding point \( Q_3 \) must fall in the contracting region, as is easily seen, and we shall show that the product \( m_2 m_3 \) is less than unity. Let \( R_2 \) and \( R_3 \) be the points where the lines \( P_2Q_2 \) and \( P_3Q_3 \) meet the curve \( y = e^{-ex} \), and we easily see that the slopes of this curve at \( R_2 \) and \( R_3 \) have greater numerical values than \( m_2 \) and \( m_3 \) respectively; and furthermore, by a little analytic geometry, we see that the product of the slopes of this curve at \( R_2 \) and \( R_3 \) is less than unity. This prod-
uct may, however approach unity as $P_3$ approaches the point where the turner cuts the last mentioned exponential curve.

Thus the integrand consists (with the possible exception of a finite last factor) of factors and pairs of factors each of which is less than unity. We wish to show that, as $n$ grows infinite, the integrand approaches zero at all points of the interval with the possible exception of one point where it cannot exceed unity. The least favorable case occurs when all the $a_i$ are equal to $e^{-s}$ and $x$ is near the value $1/e$, or else when all the $a_i$ are equal to $e^{1/e}$, and $x$ is near the value $e$. For at these points the extreme curves in fig. 1 are meeting the turner with slopes of $-1$ and $1$ respectively. But even in these cases a small segment will be indefinitely shortened by the graphical process since each of its end points must approach the point where the exponential curve meets the turner. Since the integrand may be made arbitrarily small by increasing $n$, except perhaps at one point, the value of the integral may be made arbitrarily small, and our lemma is proved.

![Diagram](image1)

![Diagram](image2)

To clarify this discussion, Fig. 6 shows the graphs of the first few margerin functions when all of the $a_i$ are equal to the lower limit $e^{-s}$, and Fig. 7 shows them when all the $a_i$ are equal to the upper limit $e^{1/e}$. One can see how the difference between the largest and smallest values of the margerin function approaches zero, although the slope at one point is $1$ or $-1$.

9. **Fundamental theorems on the convergence of infinite exponentials.**

**Theorem 6.** The infinite exponential $E_i^{x}$ converges if $m$ can be found such that
\[ e^{-e} \leq a_i \leq e^{1/e}, \quad i = m, m + 1, m + 2, \ldots \]

Stated in words the theorem says that if beyond a certain point all the exponents lie in the interval of convergence, then the infinite exponential converges. By theorem 1 it will be sufficient to prove the convergence of the \( m \)th residual \( E_{i=m}^m a_i \). By lemma 3, we can choose \( n \) so large that the total variation of \( E(a_m, a_{m+1}, \ldots, a_{m+n}, x) \) as \( x \) ranges from 0 to \( e \) shall be less than \( \epsilon \), an arbitrarily small positive number. Then by lemma 2 the value of the finite exponential \( E_{i=m}^{m+n} a_i \) will be altered by less than \( \epsilon \) if we make \( n \) larger. This proves the theorem.

Theorem 6 is somewhat analogous to the fact that an infinite product of positive factors converges when the factors are all less than or equal to unity. And the further fact that the infinite product converges even when the factors all exceed unity by small amounts which form a convergent series suggests that we investigate whether the infinite exponential may not converge when all the exponents exceed the upper limit \( e^{1/e} \) or are less than the lower limit \( e^{-e} \) by small amounts. The results were surprising in that the convergence of the "small amounts" as a series was not sufficient for the convergence of the infinite exponential.

**Theorem 7.** The infinite exponential \( E_{i=0}^m (e^{1/e} + \epsilon_i) \), where the \( \epsilon_i \) are all positive or zero, (a) will converge if

\[
\lim_{n \to \infty} \epsilon_n n^2 < \frac{e^{1/e}}{2e}
\]

(b) and will diverge if

\[
\lim_{n \to \infty} \epsilon_n n^2 > \frac{e^{1/e}}{2e}
\]

**Lemma 4.** Any properly convergent infinite exponential can be put in the form \( E(b_0^{1/b_1}, b_1^{1/b_2}, \ldots) \), which we shall call the \( b \)-form. For we need only choose each \( b_i \) equal to the \( i \)th residual.

**Lemma 5.** Any exponential in the \( b \)-form converges if each \( b_i \) is greater than unity. To prove this consider the finite exponential \( E(b_0^{1/b_1}, b_1^{1/b_2}, \ldots, b_n^{1/b_{n+1}}) \). It steadily increases with \( n \) since each exponent exceeds unity. Furthermore it never exceeds the value \( b_0 \), for if we increase the last exponent by replacing \( b_{n+1} \) by 1, the whole symbol evidently telescopes down to the value \( b_0 \).

Therefore to prove theorem 7, it is necessary and sufficient that it can be put in the \( b \)-form with each \( b_i \) greater than unity, that is, that

\[
e^{1/e} + \epsilon_n = b_n^{1/b_{n+1}}.
\]

Now since the maximum value of \( x^{1/x} \) is \( e^{1/e} \), in order that \( \epsilon_n \) be positive for all values of \( n \) it is necessary that \( b_n > b_{n+1} > e \). We accordingly write:

\[
e^{1/e} + \epsilon_n = (e + \delta_n)^{(1/e)}(e^{1/(e+\delta_{n+1})}.
\]
Expanding by Taylor's series we have
\[ \epsilon_n = e^{1/e} \left\{ \frac{1}{e^2} (\delta_n - \delta_{n+1}) - \frac{1}{2e^3} \left[ 2\delta_{n+1}(\delta_n - \delta_{n+1}) + \delta_n^2 \right] \\
+ \frac{1}{2e^4} (\delta_n - \delta_{n+1})^2 + \cdots \right\}. \]

The principal part of \( \epsilon_n \) is contained in
\[ e^{1/e} \left[ \frac{1}{e^2} (\delta_n - \delta_{n+1}) - \frac{\delta_n^2}{2e^3} \right]. \]

To make the principal part of \( \epsilon_n \) positive and of as low order as possible we must have the principal part of \( \delta_n \) equal to \( k/n \) where \( k \) is some constant. So we give \( \delta_n \) this value, and find that the principal part of \( \epsilon_n \) has a maximum of \( e^{1/e}/(2en^2) \) when \( k \) equals \( e \).

Now to prove part (a) of theorem 7 we say that an exponential in the \( b \)-form can be found as above, and its exponents will be larger than \( e^{1/e} + \epsilon_n \) so the given exponential converges by comparison. But in part (b) of theorem 7 no exponential in the \( b \)-form can be found whose exponents are as large as \( e^{1/e} + \epsilon_n \), so this one cannot be convergent.

**Theorem 8.** That the infinite exponential \( E^\infty_{i=0}(e^{-e} - \epsilon_i) \) shall converge properly, where \( \epsilon_i \) are all positive or zero, and \( \epsilon_i \geq \epsilon_{i+1} \), it is necessary that \( \lim_{i \to \infty} \epsilon_i = 0 \) and it is sufficient that \( \lim_{i \to \infty} q^i \epsilon_i = 0 \) where \( q > 1 \).

A proof of this theorem may be carried through by a careful analysis of the graphical process, and is omitted because it is lengthy and tedious.

10. When the exponents of an infinite exponential are functions of \( x \), the exponential will define a function of \( x \) for all values of \( x \) which render it convergent. We shall study one such case, namely
\[ f(x) = E(e^{k_0x}, e^{k_1x}, \ldots). \]

When \( x \) is zero this function is equal to unity. Assuming that there are other values for which the exponential converges, it can be developed formally into the following series:

\[ E(e^{k_0x}, e^{k_1x}, \ldots) = 1 + k_0x + (k_0^2 + 2k_0k_1) \frac{x^2}{2!} \\
+ (k_0^3 + 6k_0^2k_1 + 3k_0k_1^2 + 6k_0k_1k_2) \frac{x^3}{3!} \\
+ (k_0^4 + 12k_0^3k_1 + 24k_0^2k_1^2 + 4k_0k_1^3 + 24k_0k_1k_2) \frac{x^4}{4!} \]
\[ + (k_0^6 + 20k_0^4k_1 + 90k_0^2k_1^2 + 80k_0^2k_2 + 5k_0k_1^3 + 60k_0k_1k_2 + 5k_0k_1k_2 + 120k_0k_1k_2^2 + 60k_0k_1k_2k_3 + 120k_0k_1k_2k_3 + 120k_0k_1k_2k_3 + 120k_0k_1k_2k_3 + \frac{x^5}{5!} + \cdots. \]

The general expression for a term in any one of these parentheses is

\[ (\alpha_0 + \alpha_1 + \cdots + \alpha_p)! \frac{\alpha_0 \cdot \alpha_1 \cdot \cdots \cdot \alpha_{p-1}}{\alpha_0! \alpha_1! \alpha_2! \cdots \alpha_p!} k_0^{\alpha_0} k_1^{\alpha_1} k_2^{\alpha_2} \cdots k_p^{\alpha_p}. \]

If all the \( k_i \) are equal to unity we obtain

\[ E(e^x, e^x, \cdots) = 1 + x + \frac{3}{2!} x^2 + \frac{16}{3!} x^3 + \cdots + \frac{(n + 1)^{n-1}}{n!} x^n + \cdots \]

and the ratio test shows that this series converges when \(-1/e < x < 1/e\). But theorem 6 shows that the exponential converges when \(-e \leq x \leq 1/e\).

This function satisfies the transcendental equation

\[ \log y = xy \]

and may be regarded as a means of solving this equation explicitly for \( y \) in terms of \( x \). Since the ratio of each term to the preceding is always less than \( ex \), it follows that the remainder of the series beyond the \( n \)th term is less than the geometric series

\[ \frac{(n + 1)^{n-1}}{n!} x^n (1 + ex + e^2x^2 + \cdots) = \frac{(n + 1)^{n-1}}{n!} \frac{x^n}{1 - ex}. \]

11. An arbitrary function of \( x \) whose value is unity when \( x \) is zero can be developed into an infinite exponential of the type considered in article 10 by equating the terms in its series development to the terms in the development of the exponential and determining the values of the \( k_i \) one after another. For instance:

\[ \cos x = E(e^{-x^2/2}, e^{-x^2/8}, e^{-x^2/60}, \cdots) \]

\[ \sin x = xE(e^{-x^2/6}, e^{-x^2/80}, e^{-x^2/1280}, \cdots). \]

The author was keenly disappointed in not finding either simple values for the \( k_i \) or a general expression for the \( n \)th exponent, especially since these functions have such elegant expansions as infinite series and infinite products.

12. The solution for \( y \) in terms of \( x \) of the transcendental equation*

* H. L. Slobin, this MONTHLY, Oct. 1931, p. 444, solved this equation in terms of a parameter. He was interested in finding positive, rational, unequal values of \( x \) and \( y \) that satisfy the equation, one obvious pair being 2 and 4.
may be carried out by first writing it in the form
\[ y = (x^{1/z})^y \]
and then substituting the whole right side of this equation for the \( y \) that stands in the exponent, getting
\[ y = (x^{1/z})^{(x^{1/z})^y}. \]
Continued substitution gives the infinite exponential
\[ y = E(x^{1/z}, x^{1/z}, \ldots) \]
which converges for all values of \( x \) greater than \( 1/e \).

The graph of this infinite exponential has a ninety degree corner at the point \((e, e)\). The solid line in Fig. 8 is the graph of the infinite exponential while the whole curve is the graph of the original equation for positive values of \( x \) and \( y \).

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ISOGONAL AND ISOTOMIC CONJUGATES AND THEIR
PROJECTIVE GENERALIZATION

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1. Introduction. Let \( P \) be a point in the plane of the triangle \( A_1A_2A_3 \), and designate the sides of the triangle by \( a_i \) and the lines \( A_iP \) by \( p_i \). If \( p'_i \) is determined so that angle \( a_3p_1 = \text{angle } a_1p'_1 \), sense taken into account, then \( p_1 \) and \( p'_1 \) are called isogonal conjugate lines. The following fundamental incidence properties follow immediately from the trigonometric form of the theorems of Ceva and Menelaus.