MAT 771 FUNCTIONAL ANALYSIS
HOMEWORK 1 SOLUTIONS

(1) Let $X$ be the set of all bounded sequences of complex numbers
$$X = \{ (\xi_j) : \xi_j \in \mathbb{C}, \, j = 1, 2, \cdots \}.$$ For $x = (\xi_j), y = (\eta_j) \in X$, define
$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|.$$ Show $d$ is a metric on $X$.

Solution: Let $x = (\xi_j), y = (\eta_j) \in X$. Then $\exists M_1, M_2 > 0$ such that $|\xi_j| < M_1$ and $|\eta_j| < M_2$, $\forall j = 1, 2, \cdots$ $(|\xi_j|)$ and $(|\eta_j|)$ are bounded sequences in $\mathbb{R}$, so there exist $\sup_{j \in \mathbb{N}} |\xi_j|$ and $\sup_{j \in \mathbb{N}} |\eta_j|$. On the other hand, by the triangle inequality for numbers, for each $j \in \mathbb{N}$,
$$|\xi_j + \eta_j| \leq |\xi_j| + |\eta_j| \leq M_1 + M_2.$$ So $(|\xi_j + \eta_j|)$ is also a bounded sequence in $\mathbb{R}$ and hence there exists $\sup_{j \in \mathbb{N}} |\xi_j + \eta_j|$. Similarly, there exists
$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j|.$$ (M1) and (M2) are clearly satisfied. For the triangle inequality (M3), it suffices to show that
$$\sup_{j \in \mathbb{N}} |\xi_j + \eta_j| \leq \sup_{j \in \mathbb{N}} |\xi_j| + \sup_{j \in \mathbb{N}} |\eta_j|,$$ which follows from the triangle inequality for numbers and the definition of supremum.

(2) Let $X$ be the set of continuous real-valued functions defined on the closed interval $[a, b]$. Let $x, y : [a, b] \to \mathbb{R}$ be continuous and define
$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|.$$ Show that $d$ is a metric on $X$. 

Solution: Let $x(t)$ and $y(t)$ be two continuous real-valued functions on $[a, b]$, then so is $|x(t) - y(t)|$. By Max-Min Theorem, the function $|x(t) - y(t)|$ attains a maximum value on $[a, b]$, so

$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|$$

exists. (M1) and (M2) are clearly satisfied. For the triangle inequality (M3), it suffices to show that

$$\max_{t \in [a, b]} |x(t) + y(t)| \leq \max_{t \in [a, b]} |x(t)| + \max_{t \in [a, b]} |y(t)|$$

which follows from the triangle inequality for numbers and the definition of maximum.

(3) The diameter $\delta(A)$ of a nonempty set $A$ in a metric space $(X, d)$ is defined to be

$$\delta(A) = \sup_{x, y \in A} d(x, y).$$

$A$ is said to be bounded if $\delta(A) < \infty$. Show that $A \subset B$ implies $\delta(A) \leq \delta(B)$.

Solution: If $\delta(B) = \infty$, we are done since either $\delta(A) < \infty$ or $\delta(A) = \infty$. Suppose that $\delta(B) < \infty$. Then $\delta(B)$ is an upper bound of the set $\{d(x, y) : x, y \in A\} \subset \mathbb{R}$ and so there exists $\delta(A) = \sup_{x, y \in A} d(x, y)$ and $\delta(A) \leq \delta(B)$.

(4) Show that $\delta(A) = 0$ if and only if $A$ consists of a single point.

Solution: If $A$ consists of a single point, clearly $\delta(A) = 0$. Suppose $\delta(A) = 0$. Then for any $x, y \in A$,

$$d(x, y) \leq \delta(A) = 0 \implies d(x, y) = 0 \implies x = y.$$

Hence, $A$ has a single point.