MAT 771 FUNCTIONAL ANALYSIS
HOMEWORK 2

(1) Let \((X,d)\) be a metric space. Let \(x \in X\) and let \(\epsilon > 0\) be given. Show that \(B(x,\epsilon)\) is open.
(2) If \(x_0\) is an accumulation point of a set \(A \subset (X,d)\), show that any neighbourhood of \(x_0\) contains infinitely many points of \(A\).
(3) Let \((X,d)\) be a metric space and \(A \subset X\). Show that \(\overline{A}\) is the smallest closed set containing \(A\).
(4) Let \((X,d)\) be a metric space and \(A \subset X\). Show that \(x \in \overline{A}\) if and only if \(\forall\) open set \(U(x)\) in \(X\), \(U(x) \cap A \neq \emptyset\).
(5) Show that \(\overline{A \cup B} = \overline{A} \cup \overline{B}\) and \(\overline{A \cap B} \subset \overline{A} \cap \overline{B}\). Given an example that shows \(\overline{A \cap B} \neq \overline{A} \cap \overline{B}\).
(6) Let \(x = (\xi_j) \in \ell^p\) with \(1 \leq p < \infty\). Show that given \(\epsilon > 0\) there exists a positive integer \(N > 0\) such that \(\sum_{j=N+1}^{\infty} |\xi_j|^p < \epsilon\).
(7) Show that a mapping \(T : X \longrightarrow Y\) is continuous if and only if the inverse image of any closed set \(F \subset Y\) is closed in \(X\).
(8) Show that the image of an open set under a continuous mapping need not be open.