(1) If $x_0$ is an accumulation point of a set $A \subset (X, d)$, show that any neighbourhood of $x_0$ contains infinitely many points of $A$.

**Solution:** Since $x_0$ is an accumulation point of $A \subset X$, for each $n = 1, 2, \cdots$,

$$\left[B \left(x_0, \frac{1}{n}\right) \setminus \{x_0\}\right] \cap A \neq \emptyset.$$  

For each $n = 1, 2, \cdots$, choose $x_n \in \left[B \left(x_0, \frac{1}{n}\right) \setminus \{x_0\}\right] \cap A$. Given $\epsilon > 0$, by Archimedean property there exists a positive integer $N$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$d(x_n, x_0) < \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

that is, for all $n \geq N$,

$$x_n \in B(x_0, \epsilon) \cap A \subset A.$$  

In fact, we just proved that if $x_0$ is an accumulation point of $A \subset X$, then there exists a sequence $(x_n) \subset A$ such that $x_n \to x_0$.

(2) Let $(X, d)$ be a metric space and $A \subset X$. Show that $\bar{A}$ is the smallest closed set containing $A$.

**Solution:** By the definition $\bar{A} = A \cup A'$, clearly $A \subset \bar{A}$.

First we show that $\bar{A}$ is closed. Let $x \in X \setminus \bar{A}$. Since $x \notin \bar{A}$ ($x \notin A$ and $x \notin A'$), there exist an open set $U(x)$ in $X$ such that $U(x) \cap A = \emptyset$ and so $U(x) \cap A' = \emptyset$ (If $U(x)$ contains an accumulation point of $A$, $U(x) \cap A$ must be nonempty). Thus, $U(x) \cap \bar{A} = \emptyset$. This means that $U(x) \subset X \setminus \bar{A}$ i.e. $X \setminus \bar{A}$ is open. Hence, $\bar{A}$ is closed.

Next we show that $\bar{A}$ is the smallest closed set containing $A$. Let $F$ be a closed set containing $A$ and let $x \notin F$. Then
Let \( x \in X \setminus F \). Since \( X \setminus F \) is open, there exists an open neighbourhood of \( x \), \( U(x) \) (for instance \( U(x) = B(x, \epsilon) \) for some \( \epsilon > 0 \)) such that \( U(x) \subset X \setminus F \subset X \setminus A \). This implies that \( U(x) \cap A = \emptyset \). Hence, \( x \not\in A' \). Therefore, \( \tilde{A} \subset F \).

Since \( \tilde{A} \) is the smallest closed set containing \( A \), \( \tilde{A} \) can be written as

\[
\tilde{A} = \bigcap \{ F \subset X : F \text{ is closed}, A \subset F \}.
\]

(3) Let \((X,d)\) be a metric space and \( A \subset X \). Show that \( x \in \tilde{A} \) if and only if \( \forall \) open set \( U(x) \) in \( X \), \( U(x) \cap A \neq \emptyset \).

**Solution:** (\(\Rightarrow\)) Let \( x \in \tilde{A} \). Then \( x \in A \) or \( x \in A' \). Let \( U(x) \) be any open set containing \( x \). If \( x \in A \) then we are done. Suppose that \( x \not\in A \). Then \( x \in A' \) and so \( U(x) \cap A = (U(x) \setminus \{x\}) \cap A \neq \emptyset \).

(\(\Leftarrow\)) Suppose that \( \forall \) open set \( U(x) \) in \( X \), \( U(x) \cap A \neq \emptyset \). If \( x \in A \). Then we are done. If not, by the assumption \( (U(x) \setminus \{x\}) \cap A = U(x) \cap A \neq \emptyset \). So, \( x \in A' \).

(4) Show that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \) and \( \overline{A} \cap \overline{B} \subset \overline{A \cap B} \). Given an example that shows \( \overline{A \cap B} \neq \overline{A} \cap \overline{B} \).

**Solution:** First we show that if \( F_1 \) and \( F_2 \) be two closed subsets of a metric space \( X \), then \( F_1 \cup F_2 \) is also closed in \( X \). Let \( x \in X \setminus (F_1 \cup F_2) = (X \setminus F_1) \cap (X \setminus F_2) \). Then \( x \in X \setminus F_1 \) and \( x \in X \setminus F_2 \). Since both \( X \setminus F_1 \) and \( X \setminus F_2 \) are open in \( X \), there exist \( \epsilon_1, \epsilon_2 > 0 \) such that \( B(x, \epsilon_1) \subset X \setminus F_1 \) and \( B(x, \epsilon_2) \subset X \setminus F_2 \). Let \( \epsilon = \min\{\epsilon_1, \epsilon_2\} \). Then \( B(x, \epsilon) \subset X \setminus (F_1 \cup F_2) \) and so \( X \setminus (F_1 \cup F_2) \) is an open set.

\[
A \subset \overline{A}, B \subset \overline{B} \Rightarrow A \cup B \subset \overline{A} \cup \overline{B} \\
\Rightarrow \overline{A \cup B} \subset \overline{A} \cup \overline{B} \text{ (since } \overline{A} \cup \overline{B} \text{ is closed).}
\]

On the other hand,

\[
A, B \subset A \cup B \Rightarrow A, B \subset \overline{A \cup B} \\
\Rightarrow \overline{A}, \overline{B} \subset \overline{A \cup B} \\
\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}.
\]
Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$$
\begin{align*}
A \cap B \subset A, B &\implies A \cap B \subset \overline{A}, \overline{B} \\
\implies \overline{A \cap B} \subset \overline{A}, \overline{B} \\
\implies \overline{A \cap B} \subset \overline{A \cap B}.
\end{align*}
$$

Let $X$ be $\mathbb{R}$ with the usual Euclidean metric. Let $A = \left[ 0, \frac{1}{2} \right]$ and $B = \left[ \frac{1}{2}, 0 \right]$. Then $A \cap B = \emptyset$ while $\overline{A} \cap \overline{B} = \left[ 0, \frac{1}{2} \right] \cap \left[ \frac{1}{2}, 0 \right] = \left\{ \frac{1}{2} \right\}$. This example shows that it is not necessarily true that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(5) Let $x = (\xi_j) \in \ell^p$ with $1 \leq p < \infty$. Show that given $\epsilon > 0$ there exists a positive integer $N > 0$ such that $\sum_{j=N+1}^{\infty} |\xi_j|^p < \epsilon$.

**Solution:** Let us assume contrary. That is, let us assume that there exists $\epsilon > 0$ such that $\forall N = 1, 2, \ldots$, $\sum_{j=N+1}^{\infty} |\xi_j|^p > \epsilon$. Let $s = \sum_{j=1}^{\infty} |\xi_j|^p < \infty$. Then $\lim_{n \to \infty} \sum_{j=1}^{n} |\xi_j|^p = s$. So, there exists a positive integer $N'$ such that $\left| \sum_{j=1}^{n} |\xi_j|^p - s \right| < \epsilon$, $\forall n \geq N'$. That is, $s - \epsilon < \sum_{j=1}^{n} |\xi_j|^p < s + \epsilon$, $\forall n \geq N$. In particular, $s - \epsilon < \sum_{j=1}^{N'} |\xi_j|^p < s + \epsilon$. Adding to the last inequality $\sum_{j=N'+1}^{\infty} |\xi_j|^p$, we obtain

$$
\sum_{j=N'+1}^{\infty} |\xi_j|^p < s < s + \epsilon + \sum_{j=N'+1}^{\infty} |\xi_j|^p.
$$
This is a contradiction since \( \sum_{j=N'+1}^{\infty} |\xi_j|^p > \epsilon \).

(6) Show that a mapping \( T : X \rightarrow Y \) is continuous if and only if the inverse image of any closed set \( F \subset Y \) is closed in \( X \).

**Solution:** (\( \Rightarrow \)) Suppose that \( T : X \rightarrow Y \) is continuous. Let \( F \) be a closed set in \( Y \). Then \( Y \setminus F \) is open in \( Y \). Since \( T \) is continuous, \( T^{-1}(Y \setminus F) = X \setminus T^{-1}(F) \) is open in \( X \) i.e. \( T^{-1}(F) \) is closed in \( X \).

(\( \Leftarrow \)) Suppose that the inverse image of any closed set \( F \subset Y \) is closed in \( X \). Let \( U \) be an open set in \( Y \). Then \( Y \setminus U \) is closed in \( Y \). So by the assumption, \( T^{-1}(Y \setminus U) = X \setminus T^{-1}(U) \) is closed in \( X \) i.e \( T^{-1}(U) \) is open in \( X \). Since the choice of \( U \) was arbitrary, \( T \) is continuous.

(7) Show that the image of an open set under a continuous mapping need not be open.

**Solution:** Let \( X \) be \( \mathbb{R} \) with the usual Euclidean metric. The function \( f : X \rightarrow X \) defined by \( f(x) = x^2 \) is continuous. \((-1, 1)\) is open in \( X \) but \( f(-1, 1) = [0, 1) \) is not open in \( X \).