(1) Let $X$ be the vector space of all complex $2 \times 2$ matrices and define $T : X \to X$ by $Tx = bx$, where $b \in X$ is fixed and $bx$ denotes the usual matrix multiplication. Show that $T$ is linear. Under what condition does $T^{-1}$ exist?

(2) Let $T : \mathcal{D}(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1, \cdots, x_n\}$ is a linearly independent set in $\mathcal{D}(T)$, show that the set $\{Tx_1, \cdots, Tx_n\}$ is linearly independent.

(3) Let $T : X \to Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $\mathcal{R}(T) = Y$ if and only if $T^{-1}$ exists.

(4) Let $X$ and $Y$ be normed spaces. Show that a linear operator $T : X \to Y$ is bounded if and only if $T$ maps bounded sets in $X$ into bounded sets in $Y$.

(5) If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$. 

(6) Define an operator $T : \ell^\infty \to \ell^\infty$ by 
\[
T(\xi_j) = \left( \frac{\xi_j}{j} \right)
\]
for each $(\xi_j) \in \ell^\infty$. Show that $T$ is linear and bounded.

(7) Let $T$ be a bounded linear operator from a normed space $X$ onto a normed space $Y$. Suppose that there is a positive $b$ such that 
$\|Tx\| \geq b\|x\|$ 
for all $x \in X$. Show that $T^{-1} : Y \to X$ exists and bounded.