(1) Let $X$ be the vector space of all complex $2 \times 2$ matrices and define $T : X \rightarrow X$ by $Tx = bx$, where $b \in X$ is fixed and $bx$ denotes the usual matrix multiplication. Show that $T$ is linear. Under what condition does $T^{-1}$ exist? 

**Solution:** Let $c$ and $d$ be scalars and $x, y \in X$. Then

$$T(cx + dy) = b(cx + dy)$$
$$= b(cx) + b(dy) \text{ (distributive law)}$$
$$= c(bx) + d(by)$$
$$= cTx + dTy.$$ 

Hence, $T$ is linear. $T^{-1}$ exists if $b$ is invertible or equivalently $\det b \neq 0$.

(2) Let $T : \mathcal{D}(T) \rightarrow Y$ be a linear operator whose inverse exists. If $\{x_1, \ldots, x_n\}$ is a linearly independent set in $\mathcal{D}(T)$, show that the set $\{Tx_1, \ldots, Tx_n\}$ is linearly independent. 

**Solution:** Suppose that $c_1T(x_1) + \cdots + c_nT(x_n) = 0$. Since $T$ is linear,

$$c_1T(x_1) + \cdots + c_nT(x_n) = T(c_1x_1 + \cdots + c_nx_n)$$
$$= 0.$$ 

Since $T^{-1}$ exists, $T$ is one-to-one. So, $c_1x_1 + \cdots + c_nx_n = 0$. Since $x_1, \ldots, x_n$ are linearly independent, we have

$$c_1 = c_2 = \cdots = c_n = 0.$$ 

Hence, $Tx_1, \ldots, Tx_n$ are linearly independent.

(3) Let $T : X \rightarrow Y$ be a linear operator and $\dim X = \dim Y = n < \infty$. Show that $\mathcal{R}(T) = Y$ if and only if $T^{-1}$ exists. 

**Solution:** It follows from the formula

$$\dim \mathcal{D}(T) = \dim \mathcal{N}(T) + \dim \mathcal{R}(T)$$
that you learned in linear algebra.

(4) Let $X$ and $Y$ be normed spaces. Show that a linear operator $T : X \rightarrow Y$ is bounded if and only if $T$ maps bounded sets in $X$ into bounded sets in $Y$.

**Solution:** Let $A \subset X$ be a bounded set. Then there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$. Suppose that a linear operator $T : X \rightarrow Y$ is bounded. Then there exists a real number $c > 0$ such that

$$\|Tx\| \leq c\|x\|$$

for all $x \in X$. Since $\|x\| \leq M$ for all $x \in A$,

$$\|Tx\| \leq cM$$

for all $x \in A$. So, $R(T_A)$ is bounded. Conversely, assume that $T$ maps bounded sets in $X$ into bounded sets in $Y$. Let $A = \{x \in X : \|x\| = 1\}$. Then clearly $A$ is bounded and by the assumption $R(T_A)$ is bounded i.e. there exist a real number $M > 0$ such that $\|y\| \leq M$ for all $y \in R(T_A)$. If $x = O$, clearly $\|Tx\| \leq M\|x\|$. Suppose that $x \neq O$. Then $\frac{x}{\|x\|} \in A$ and so,

$$\frac{1}{\|x\|}\|Tx\| = \left\| T \frac{x}{\|x\|} \right\| \leq M.$$

That is, $\|Tx\| \leq M\|x\|$. Therefore, $T$ is bounded.

(5) If $T \neq O$ is a bounded linear operator, show that for any $x \in D(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|\|x\|$.

**Solution:** Let $x \in D(T)$ such that $\|x\| < 1$. If $Tx = O$ then clearly $\|Tx\| < \|T\|$ since $T \neq O$. Suppose that $Tx \neq O$. Then

$$\|Tx\|\|x\| < \|Tx\| \Rightarrow \|Tx\| < \frac{\|Tx\|}{\|x\|} \leq \sup_{y \in D, y \neq 0} \frac{\|Ty\|}{\|y\|} = \|T\|.$$
Hence, $\|Tx\| < \|T\|$. 

(6) Define an operator $T : \ell^\infty \rightarrow \ell^\infty$ by 

$$T(\xi_j) = \left( \frac{\xi_j}{j} \right)$$

for each $(\xi_j) \in \ell^\infty$. Show that $T$ is linear and bounded. 

**Solution:** Let $a, b$ be complex numbers and $(\xi_j), (\eta_j) \in \ell^\infty$. Then 

$$T(a\xi_j + b\eta_j) = T(a\xi_j + b\eta_j)$$

$$= \left( \frac{a\xi_j + b\eta_j}{j} \right)$$

$$= a \left( \frac{\xi_j}{j} \right) + b \left( \frac{\eta_j}{j} \right)$$

$$= aT(\xi_j) + bT(\eta_j).$$

Hence $T$ is linear. Suppose that $(\xi_j) \neq (0) \in \ell^\infty$. Then 

$$\|T(\xi_j)\| = \left\| \left( \frac{\xi_j}{j} \right) \right\|$$

$$= \sup_{j \in \mathbb{N}} \left| \frac{\xi_j}{j} \right|$$

$$\leq \sup_{j \in \mathbb{N}} |\xi_j|$$

$$= \|(\xi_j)\|.$$ 

Hence $T$ is bounded. 

(7) Let $T$ be a bounded linear operator from a normed space $X$ onto a normed space $Y$. Suppose that there is a positive $b$ such that 

$$\|Tx\| \geq b\|x\|$$

for all $x \in X$. Show that $T^{-1} : Y \rightarrow X$ exists and bounded. 

**Solution:** Suppose that $Tx = 0$. Then it follows from $\|Tx\| \geq b\|x\|$ that $\|x\| \leq 0$. On the other hand, $\|x\| \geq 0$. So, $\|x\| = 0$ i.e. $x = O$. This means that $\mathcal{N}(T) = \{O\}$ and
so $T$ is one-to-one. Since $T$ is also onto, $T$ is invertible. For all $y \in Y$,

$$||y|| = ||T(T^{-1}y)|| \leq b||T^{-1}y||,$$

i.e. for all $y \in Y$,

$$||T^{-1}y|| \leq \frac{1}{b}||y||.$$

Hence, $T^{-1}$ is bounded.