Nim∞

or, From the ridiculous to the sublime is but a step.
(with apologies to Napoleon)

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1 Background

2 The Hilbert-Dickson Game

3 Nim

4 Nim$^\infty$?

5 Conclusion
§1. Background
My field, in portraits

Euclid + Gauß = Buchberger
My field, in diagrams

Euclidean algorithm

Gauss-Jordan reduction

Buchberger’s algorithm

Background

The Hilbert-Dickson Game

Nim

Nim∞

Conclusion
Recent contributions

Background
The Hilbert-Dickson Game
Nim
Nim∞?
Conclusion

primitive $S$-irreducible polynomials
(w/ Alberto Arri, Google Corp.)

border vectors
(w/ Massimo Caboara, Università di Pisa)
Animated commutative algebra

• Ideal
  • generators
  • absorption property

• Quotient ring

• Noetherian
  • Dickson’s Lemma
  • Hilbert Basis Theorem

• Hilbert function

How to communicate this to students?
§2. The Hilbert-Dickson Game
Shall we play a game?

Rules (v. 1)

- Move: choose \((x, y)\)
  - lattice point
  - not northeast of prior move (gray)
- Last move loses
Shall we play a game?

Rules (v. 1)
- Move: choose \((x, y)\)
  - lattice point
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Gameplay
- Too easy: choose \((1, 1)\)
- Reflect opponent’s choices, force into a corner
- Demonstratio in tabula
Shall we play a game?

Rules (v. 3)

• Choose several lattice points, $G$
• Move: choose $(x, y)$
  • lattice point
  • not northeast of prior move (gray)
  • not southwest of point in $G$ (red)
• last move wins
Rules (v. 2)

- Choose several lattice points, \( G \)
- For each \( d \in \mathbb{N} \), count # of points not southwest of \( G \)
  - call this \( H(d) \)
- Choose lattice point \( (x, y) \):
  - not northeast of previously-chosen point
  - For each \( d \), must leave \( H(d) \) points southwest of choices
Commutative algebra in action!

Rules?

1. not northeast of previously-chosen point?
   - Ascending Chain Condition (Noetherian)
   - Dickson’s Lemma / Hilbert Basis Theorem
Commutative algebra in action!

Rules?

v. 1 not northeast of previously-chosen point?
  - Ascending Chain Condition (Noetherian)
  - Dickson’s Lemma / Hilbert Basis Theorem

v. 3 not southwest of point in $G$?
  - move in ideal
Commutative algebra in action!

Rules?

v. 1 not northeast of previously-chosen point?
  - Ascending Chain Condition (Noetherian)
    - Dickson’s Lemma / Hilbert Basis Theorem

v. 3 not southwest of point in $G$?
  - move in ideal

v. 2 leave $H(d)$ points southwest of choices?
  - $H(d)$: Hilbert function
  - “invariant” of ideal
  - compute basis wrt different ordering
“Hilbert-Dickson” Game

let’s try it!
§3. Nim
Rules

- three rows of sticks
  - usually 7, 5, 3
- can take any number of sticks from one row
- winner takes last sticks
A 10 year-old’s introduction to Nim

Developer: middle school teacher as aid to teach binary system

instructions

high-definition graphics in 1978
(that rectangle is an explosion)
A 10 year-old’s introduction to Nim

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(in a practice session, so I didn’t win a prize)
A 10 year-old’s introduction to Nim

instructions

high-definition graphics in 1978
(that rectangle is an explosion)

Developer: middle school teacher as aid to teach binary system
(failed on me, though I did beat the game once)
(in a practice session, so I didn’t win a prize)

(Consolation: no one else won)
Theorem (Bouton)

Let $n$ be the nimber corresponding to the current heaps. If $n = 0$, next player loses.
Mathematical aspects of Nim

Theorem (Bouton)

Let $n$ be the nimber corresponding to the current heaps. If $n = 0$, next player loses.

“nimber” is not a typo
Basic idea:

- win by taking last sticks
- can sometimes “undo” opponent’s “do”
- **goal:** leave “0” sticks
  - modulo some “undo”s

Nimbers: Nim heaps

Solution: base-2 (xor) arithmetic
Nimbers: Nim heaps

Basic idea:

- win by taking last sticks
- can sometimes “undo” opponent’s “do”
- **goal**: leave “0” sticks
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*Solution: base-2 (xor) arithmetic*
Examples

Nim

Nim

Nim∞

Conclusion

Background

The Hilbert-Dickson Game

(1 ⊕ 2) ⊕ (1 ⊕ 4) ⊕ (1 ⊕ 2 ⊕ 4) = 1

to get 0, take 1 piece!

Try w/ real game
Examples

(1 \oplus 2) \oplus (1 \oplus 4) \oplus (1 \oplus 2 \oplus 4) = 1

to get 0, take 1 piece!
Examples

(1 \oplus 2) \oplus (1 \oplus 4) \oplus (1 \oplus 2 \oplus 4) = 1

to get 0, take 1 piece!

Try w/real game
In what sick world is this mathematics?

You may have thought that mathematics was a pretty serious business, and a herd of cows rampaging through a maze, watched by a gang of engineers who are either building the maze or demolishing it, lacks the proper gravitas. But, as I’ve said many times now, ‘serious’ need not equate to ‘solemn’.

― Ian Stewart
*Cows in the Maze*
O ye of little faith!

- John Conway (Princeton), *On Numbers and Games*, 1976
- *Winning Ways for Your Mathematical Plays*, 1982
  - Elwyn Berlekamp (UC Berkeley)
  - John Conway (Princeton)
  - Richard Guy (Erdős number 1)
Sprague-Grundy Theorem

Every “impartial game” is equivalent to a nimber.
More mathematical aspects of Nim

Sprague-Grundy Theorem

Every “impartial game” is equivalent to a nimber.

“impartial game”???
Shall we play a game?

game

- two players, alternating turns
- deterministic (no dice)
- transparent information (no cards)
- someone must win in finite time
- **impartial**
  - all moves, rewards available to either player
  - only difference b/w players is who goes first
- **partizan**
  - different players have different choices of move
Examples

Impartial games
- Nim (duh)
- Chompo
- Kayles
- Sprouts
- “poset games”

Partizan games
- Chess
- Go
- Hackenbush
Partizan games $\implies$ “Surreal numbers” (Conway, Knuth)

- $\{ \text{# moves after green moves} \mid \text{# moves after blue moves} \}$
Partizan games $\implies$ “Surreal numbers” (Conway, Knuth)

- $\{\text{# moves after green moves} \mid \text{# moves after blue moves}\}$
- $\{\mid\} = \text{“0”}$
- new “numbers”? let $a < b$
  - $\{a \mid b\}$ is “simplest” number “between” $a, b$
    - “simplest”? technical details. don’t ask.
  - $a, b$ can be lists of numbers
    - $\max(a) \leq \min(b)$
Numbers and games

Partizan games $\implies$ “Surreal numbers” (Conway, Knuth)

- $\{\# \text{ moves after green moves} | \# \text{ moves after blue moves}\}$

- $\{\emptyset\} = \emptyset$

- new “numbers”? let $a < b$

  - $\{a | b\}$ is “simplest” number “between” $a, b$

  “simplest”? technical details. don’t ask.

- $a, b$ can be lists of numbers

  - $\max(a) \leq \min(b)$

Conway: green = Left, blue = Right

“We favor Left”
Examples

\[ \mathbb{Z}, \text{ of course} \]

- \{0 \} = 1
- \{0 |\} = -1
- \{1 \} = 2, \{2 \} = 3, \text{ etc.}

other powers of 2
- \{0 | 1\} = -\frac{1}{2}, \{1 \frac{1}{4}, 2\} = 1 \frac{1}{2}

“left is positive”
Examples

\[ \mathbb{Z}, \text{ of course} \]

- \( \{0 \mid\} = 1 \) “left is positive”
- \( \{\mid 0\} = -1 \)
- \( \{1 \mid\} = 2, \{2 \mid\} = 3, \text{ etc.} \)

other powers of 2

- \( \{0 \mid 1\} = -\frac{1}{2}, \{1\frac{1}{4}, 2\} = 1\frac{1}{2} \)

things that make you go, “\( \omega \)hoah” and “\( \varepsilon \)ek”

- \( \{1, 2, \ldots \mid\} = \omega \)hoah! — an “infinite” number
- \( \{\mid -1, -\frac{1}{2}, -\frac{1}{4}, \ldots \} = -\varepsilon \)ek! a negative “infinitesimal”
Examples

\(\mathbb{Z}\), of course

- \(\{0\mid\} = 1\)  
- \(\{\mid 0\} = -1\)
- \(\{1\mid\} = 2, \{2\mid\} = 3\), etc.

Other powers of 2

- \(\{0\mid 1\} = -\frac{1}{2}, \{\frac{1}{4}, 2\} = 1\frac{1}{2}\)

Things that make you go, “\(\omega\)hoah” and “\(\epsilon\)ek”

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But is it useful?

Perfectly useful arithmetic —

— and largest possible ordered field!
Nim impartial $\iff$ all moves available to either player

- $\{a \mid a\} = *a$, a “nimber”
  - $*a \oplus *b = \text{mex}(\{c + b : c < a\} \cup \{a + d : d < b\})$
Nim and numbers

Nim impartial $\iff$ all moves available to either player

- $\{a | a\} = *a$, a “nimber”
- $*a \oplus *b = \text{mex}(\{c + b : c < a\} \cup \{a + d : d < b\})$

Mex rule

$\{a | b\}$ minimal, simplest* excluded number b/\ w a, b

*still a technical detail — don’t ask
§4. $\text{Nim}^\infty$?
“Hilbert-Dickson”: impartial game...

- two alternating players
- deterministic
- transparent information
- win in finite time

impartial?
- all moves, rewards available to either player
- only difference b/w players is who goes first

Notice Sprague-Grundy applies!
“Hilbert-Dickson”: impartial game...

- two alternating players
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impartial?
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Notice
Sprague-Grundy applies!
similarities

- rows
- each move removes “sticks”
- analyze w/ nimbers
...variant of Nim...

similarities

- rows
- each move removes “sticks”
- analyze w/ nimbers

Differences

- infinitely many rows
- affects multiple rows
- $\omega$hoah!
- forbidden positions
  - choices change
  - challenge
WWFYMP describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
  - same 2d board as Hilbert-Dickson
  - \( n \) movable pieces
    - move left in same row, or
    - down in any column
  - winner moves last piece

... known variant?
WWFYMP describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
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    - move left in same row, or
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- 3d Nim, 4d Nim, …
- “Anyone for Hilbert Nim?”
**WWFYMP** describes many variants of Nim

- 2d Nim similar to Hilbert-Dickson
  - same 2d board as Hilbert-Dickson
  - $n$ movable pieces
    - move left in same row, or
    - down in any column
  - winner moves last piece
- 3d Nim, 4d Nim, . . .
- “Anyone for Hilbert Nim?”

No variant equivalent to “Hilbert-Dickson game”
Means justify ends!

Hilbert-Dickson appears to be new
§5. Conclusion
Not bad for a day's few months' work

I get paid to play games!
Not bad for a day’s few months’ work

I get paid to play games!

er... ahem...

Commutative algebra
- ideals, absorption, Noetherian rings, Dickson’s Lemma, Hilbert Basis Theorem, ...

Combinatorial game theory
- Nim, nimbers, binary, mex, ...
Yet another variant

Gröbner basis in $\mathbb{F}_2[x,y]$

- gameboard? same as Hilbert-Dickson game
- polynomial $\leadsto$ monomials $\leadsto$ pieces
  - each polynomials has “distinguished” monomial
- moves?
  - select 2 polynomials $\leadsto$ shift to cancel distinguished monomials
  - reduce result
- end with “forbidden zone” of Hilbert-Dickson game
Yet another variant

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$\therefore$ computing GB is “variant” of “Nim”
An animation is worth 1,000,000 words
Questions

mex: a strategy to compute GB’s?

**Normal** pick *smallest uncomputed* pair

**Signature-based** pick *smallest excluded* signature

**Involutive** pick *smallest uncomputed* extension

∃ **Nim-based** strategy?
The end

Fine

Finis

конец