QUIZ SOLUTIONS

QUIZ THE FIRST

1. Given the graph of \( f(x) \) at right, find \( \lim_{x \to 2^-} f(x) \), \( \lim_{x \to 2^+} f(x) \), and \( \lim_{x \to 2} f(x) \).

   \[ \text{Solution: } \lim_{x \to 2^-} f(x) = 3, \lim_{x \to 2^+} f(x) = 1. \text{ Since the limits disagree, } \lim_{x \to 2} f(x) \text{ is undefined.} \]

2. Evaluate \( \lim_{x \to 0} (4x^2 - 3x + 4) \).

   \[ \text{Solution: Since } 4x^2 - 3x + 4 \text{ is a polynomial, and therefore continuous, we can evaluate the limit by substitution. Thus, } \lim_{x \to 0} (4x^2 - 3x + 4) = 4 \cdot 0^2 - 3 \cdot 0 + 4 = 4. \]

3. Which function(s) is (are) continuous everywhere?

   \[ 4x^2 - 3x + 4 \quad -12 \quad \frac{3}{x} \quad e^x \]

   \[ \text{Solution: Polynomials are continuous everywhere, so the first two functions are continuous. (Constants are also polynomials.) The third function has division by 0 at } x = 0, \text{ so it cannot be continuous. Using the intuitive definition of continuity, we can determine that } e^x \text{ is also continuous, because we never have to pick up the pencil while drawing its graph.} \]

4. What was §2.3 about?

   \[ \text{Solution: Shortcuts for taking derivatives.} \quad \text{— Or, derivative properties.} \quad \text{— Or, Marginal Cost and its ilk.} \]
1. By imagining tangent lines at \( P_1, P_2, \) and \( P_3 \) in the graph, state whether the slopes are positive, negative, or zero.

Solution: Slope is negative at \( P_1, \) zero at \( P_2, \) positive at \( P_3. \)

2. For \( f(x) = x^2 - 2x, \) find the average rate of change between the given points. Then, indicate the value the average rates of change are approaching.

(a) \( x = 0, \) \( x = 2 \)
(b) \( x = 0, \) \( x = 1 \)
(c) \( x = 0, \) \( x = \frac{1}{2} \)
(d) \( x = 0, \) \( x = \frac{1}{10} \)

Solution:

(a) \( \frac{\Delta y}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{(2^2 - 2 \cdot 2) - (0^2 - 2 \cdot 0)}{2} = \frac{0 - 0}{2} = 0 \)

(b) \( \frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1 - 0} = \frac{(1^2 - 2 \cdot 1) - (0^2 - 2 \cdot 0)}{1} = \frac{-1 - 0}{1} = -1 \)

(c) \( \frac{\Delta y}{\Delta x} = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = \frac{\left(\left(\frac{1}{2}\right)^2 - 2 \cdot \frac{1}{2}\right) - (0^2 - 2 \cdot 0)}{\frac{1}{2}} = \frac{-\frac{3}{4} - 0}{\frac{1}{2}} = -1.5 \)

(d) \( \frac{\Delta y}{\Delta x} = \frac{f(\frac{1}{10}) - f(0)}{\frac{1}{10} - 0} = \frac{\left(\left(\frac{1}{10}\right)^2 - 2 \cdot \frac{1}{10}\right) - (0^2 - 2 \cdot 0)}{\frac{1}{10}} = \frac{\frac{19}{100}}{\frac{1}{10}} = -1.9 \)

The values approach -2.

3. Use the definition of the instantaneous rate of change to find \( f'(0), \) where \( f(x) = x^2 - 2x. \)

Solution: First we need to find \( f'(x): \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\left(x + h\right)^2 - 2\left(x + h\right)\right) - \left(x^2 - 2x\right)}{h}
\]

\[
= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 - 2x - 2h\right) - \left(x^2 - 2x\right)}{h} \quad \text{expand } (x + h)^2
\]

\[
= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} \quad \text{distribution}
\]

\[
= \lim_{h \to 0} \frac{h(2x + h - 2)}{h} \quad \text{factoring}
\]

\[
= \lim_{h \to 0} (2x + h - 2)
\]

\[
= 2x + 0 - 2.
\]

Since \( f'(x) = 2x - 2, \) we can find \( f'(0) = 2 \cdot 0 - 2 = -2. \)
1. Find the derivative of \( \frac{1}{64}x^8 - \frac{1}{16}x^4 + 2x + 1 \).

   **Solution:** Use the properties:

   (1) \( \frac{d}{dx} \left( \frac{1}{64}x^8 - \frac{1}{16}x^4 + 2x + 1 \right) = \left( \frac{d}{dx} \frac{1}{64}x^8 \right) - \left( \frac{d}{dx} \frac{1}{16}x^4 \right) + \left( \frac{d}{dx} 2x \right) + \left( \frac{d}{dx} 1 \right) \)

   (2) \( = \frac{1}{64} \left( 8x^7 \right) - \frac{1}{16} \left( 4x^3 \right) + 2 \left( \frac{d}{dx} x \right) + \left( \frac{d}{dx} 1 \right) \)

   (3) \( = \frac{1}{64} (8x^7) - \frac{1}{16} (4x^3) + 2(1) + (0) \)

   (4) \( = \frac{1}{8}x^7 + \frac{1}{4}x^3 + 2. \)

   We used

   (1) the derivative of a sum is the sum of the terms’ derivatives;
   (2) the derivative of a constant multiple is the constant multiple of the derivative;
   (3) the derivative of \( x^n \) is \( nx^{n-1} \); the derivative of \( x \) is 1; and the derivative of 1 is 0;
   (4) “obvious” simplifications.

2. If \( f(x) = \frac{1}{64}x^8 - \frac{1}{16}x^4 + 2x + 1 \), find \( f’(2) \).

   **Solution:** We already found the derivative in the first problem, so we need merely substitute \( x = 2 \):

   \( f’(2) = \frac{1}{8} \cdot 2^7 - \frac{1}{4} \cdot 2^3 + 2 = 16 - 2 + 2 = 16. \)

3. A company sells widgets at a total cost of approximately \( C(x) = 32x^{3/4} \) dollars for \( x \) widgets. Find the marginal cost of 10,000 widgets, and interpret your answer.

   **Solution:** Marginal cost is approximately equal to the derivative, so we compute

   \( MC(x) = \frac{d}{dx} (32x^{3/4}) = 32 \frac{d}{dx} (x^{3/4}) = 32 \cdot \frac{3}{4} x^{3/4-1} = 24x^{-1/4} = \frac{24}{x^{1/4}}. \)

   When \( x = 10,000 = 10^4 \), we have

   \( MC(10^4) = \frac{24}{(10^4)^{1/4}} = \frac{24}{10} = 2.4. \)

4. Use the intuitive definition of the derivative to explain why the derivative of the function \( f(x) = -3x \) must be \( f'(x) = -3. \)

   *(Hint: By “intuitive”, I refer to the geometric interpretation of the derivative.)*

   **Solution:** The function \( f(x) = -3x \) is a line. For any \( x \), the line tangent to \( f \) at \( x \) will coincide with \( f \), because the only line tangent to a line is the line itself. The derivative is the slope of the tangent line, hence the slope of \( f \), which is -3.
5. A company can produce LCD digital alarm clocks at a cost of \( C(x) = 3x - 1000 \) dollars for \( x \) clocks.

(a) Find the average cost function, \( AC(x) \).

\[ \text{Solution:} \] We find the average by dividing a total by the number of objects involved in computing that total, so \( AC(x) = \frac{C(x)}{x} = \frac{3x - 1000}{x} \). You \textit{can} simplify this to \( AC(x) = 3 - \frac{1000}{x} \) or even \( AC(x) = 3 - 1000x^{-1} \), but you don’t need to.

(b) Find the marginal average cost function, \( MAC(x) \).

\[ \text{Solution:} \] The second formulation makes it easiest to compute the derivative, since

\[ MAC(x) = AC'(x) = 0 - 1000(-x^{-2}) = \frac{1000}{x^2}. \]

The first formulation requires the quotient rule:

\[ MAC(x) = AC'(x) = \frac{(3-0) - (3x - 1000) \cdot 1}{x^2} = \frac{3x - (3x - 1000)}{x^2} = \frac{1000}{x^2}. \]

(c) Evaluate \( MAC(x) \) at \( x = 350 \), rounded to the nearest tenth of a cent.

\[ \text{Solution:} MAC(350) = \frac{1000}{350^2} = 0.008. \] (When measuring in dollars, “cents” appear in the second decimal place, so “tenths of a cent” appear in the third.)