**Bonus Problem 11:**

Consider the angle $\psi$ between the radius vector and the tangent line to a curve, $r = f(\theta)$, given in polar coordinates, as shown in Fig. 1. Show that $\psi = \tan^{-1}(r/(dr/d\theta))$.

Figure 1: The tangent line to the curve $r = f(\theta)$ makes an angle of $\psi$ with respect to the radial line at the point of tangency, and an angle $\phi$ with respect to the $x$-axis.

**Proof:**

- Consider $\phi = \theta + \psi$. Then $r = f(\theta)$ is given in polar coordinates by
  
  $$
  x = r \cos \theta, \quad y = r \sin \theta, 
  $$
  
  with associated derivatives given by.

  $$
  \frac{dx}{d\theta} = -r \sin \theta + \cos \theta \frac{dr}{d\theta}, \quad \frac{dy}{d\theta} = r \cos \theta + \sin \theta \frac{dr}{d\theta}. 
  $$

- From the geometry of the problem in which it is evident that $2\pi - \psi - \theta = 2\pi - \phi$, we have that $\psi = \phi - \theta$, and consequently, using a familiar multiple angle formula from trigonometry, that
  
  $$
  \tan \psi = \tan(\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta}. 
  $$

- Since the tangent line to the curve $f(\theta)$ makes an angle $\phi$ with respect to the $x$-axis we have that $\tan \phi = \frac{dy/d\theta}{dx/d\theta}$, and trivially from the geometry that $\tan \theta = y/x$. Substituting these into (3) gives,

  $$
  \tan \psi = \frac{\frac{dy}{d\theta} - \frac{y}{x}}{1 + \left(\frac{y}{x}\right) \frac{dy}{d\theta}} = \frac{x \frac{dy}{d\theta} - y \frac{dx}{d\theta}}{x \frac{dx}{d\theta} + y \frac{dy}{d\theta}}
  $$

  $$
  = \frac{x \left(r \cos \theta + \sin \theta \frac{dr}{d\theta}\right) - y \left(-r \sin \theta + \cos \theta \frac{dr}{d\theta}\right)}{x \left(-r \sin \theta + \cos \theta \frac{dr}{d\theta}\right) + y \left(r \cos \theta + \sin \theta \frac{dr}{d\theta}\right)}
  $$

  $$
  = \tan \left(\frac{\phi - \theta}{1 + \theta \tan \theta}\right). 
  $$
Substituting (1) into (5) yields,

\[
\tan \psi = \frac{r \cos \theta (r \cos \theta + \sin \theta (dr/d\theta)) - r \sin \theta (-r \sin \theta + \cos \theta (dr/d\theta))}{r \cos \theta (-r \sin \theta + \cos \theta (dr/d\theta)) + r \sin \theta (r \cos \theta + \sin \theta (dr/d\theta))}
\]

(6)

\[
= \frac{r^2}{rdr/d\theta}
\]

(7)

\[
= \frac{r}{dr/d\theta}
\]

(8)

which is the desired result.