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**Lecture 17 Notes**

These notes correspond to Section 10.1 in the text.

## Three-Dimensional Coordinate Systems

Over the course of the next several lectures, we will learn how to work with locations and directions in three-dimensional space, in order to easily describe objects such as lines, planes and curves. This will set the stage for the study of functions of two variables, the graphs of which are surfaces in space.

### Points in Three-Dimensional Space

Previously, we have identified a point in the  $xy$ -plane by an ordered pair that consists of two real numbers, an  $x$ -coordinate and  $y$ -coordinate, which denote signed distances along the  $x$ -axis and  $y$ -axis, respectively, from the origin, which is the point  $(0, 0)$ . These axes, which are collectively referred to as the coordinate axes, divided the plane into four quadrants.

We now generalize these concepts to three-dimensional space, or  $xyz$ -space. In this space, a point is represented by an *ordered triple*  $(x, y, z)$  that consists of three numbers, an  $x$ -coordinate, a  $y$ -coordinate, and a  $z$ -coordinate. As in the two-dimensional  $xy$ -plane, these coordinates indicate the signed distance along the *coordinate axes*, the  $x$ -axis,  $y$ -axis and  $z$ -axis, respectively, from the origin, denoted by  $O$ , which has coordinates  $(0, 0, 0)$ . There is a one-to-one correspondence between a point in  $xyz$ -space and a triple in  $\mathbb{R}^3$ , which is the set of all ordered triples of real numbers. This correspondence is known as a *three-dimensional rectangular coordinate system*.

**Example** Figure 1 displays the point  $(2, 3, 1)$  in  $xyz$ -space, denoted by the letter  $P$ , along with its projections onto the coordinate planes (described below). The origin is denoted by the letter  $O$ .  $\square$

### Planes in Three-Dimensional Space

Unlike two-dimensional space, which consists of a single plane, the  $xy$ -plane, three-dimensional space contains infinitely many planes, just as two-dimensional space consists of infinitely many lines. Three planes are of particular importance: the  $xy$ -plane, which contains the  $x$ - and  $y$ -axes; the  $yz$ -plane, which contains the  $y$ - and  $z$ -axes; and the  $xz$ -plane, which contains the  $x$ - and  $z$ -axes.

Alternatively, the  $xy$ -plane can be described as the set of all points  $(x, y, z)$  for which  $z = 0$ . Similarly, the  $yz$ -plane is the set of all points of the form  $(0, y, z)$ , while the  $xz$ -plane is the set of all points of the form  $(x, 0, z)$ .

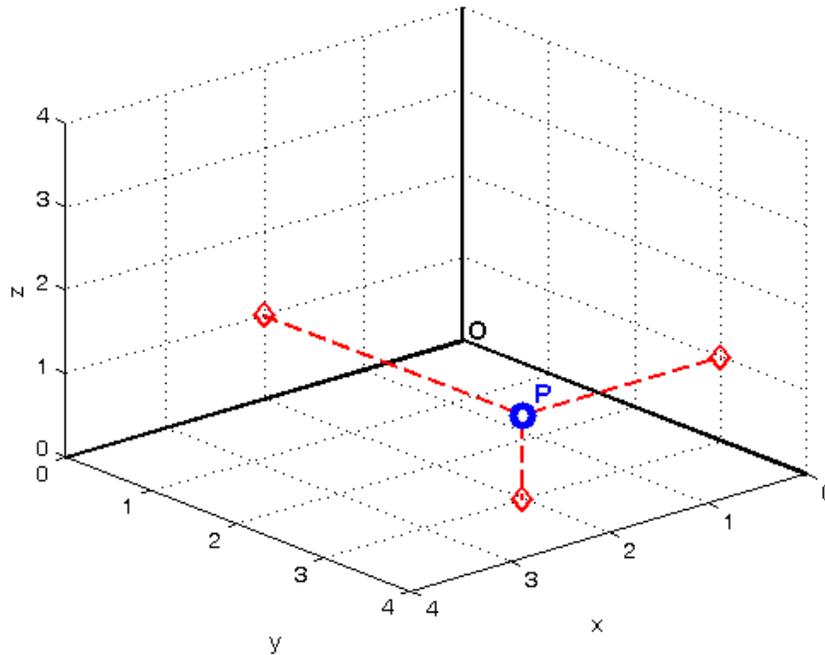


Figure 1: The point  $(2, 3, 1)$  in  $xyz$ -space, denoted by the letter  $P$ . The origin is denoted by the letter  $O$ . The projections of  $P$  onto the coordinate planes are indicated by the diamonds. The dashed lines are line segments perpendicular to the coordinate planes that connect  $P$  to its projections.

Just as the  $x$ -axis and  $y$ -axis divide the  $xy$ -plane into four quadrants, these three planes divide  $xyz$ -space into eight *octants*. Within each octant, all  $x$ -coordinates have the same sign, as do all  $y$ -coordinates, and all  $z$ -coordinates. In particular, the *first octant* is the octant in which all three coordinates are positive.

### Plotting Points in $xyz$ -space

Graphing in  $xyz$ -space can be difficult because, unlike graphing in the  $xy$ -plane, depth perception is required. To simplify plotting of points, one can make use of *projections* onto the coordinate planes. The projection of a point  $(x, y, z)$  onto the  $xy$ -plane is obtained by connecting the point to the  $xy$ -plane by a line segment that is perpendicular to the plane, and computing the intersection of the line segment with the plane.

Later, we will learn more about how to compute projections of points onto planes, but in this

relatively simple case, it follows from our working definition that the projection of the point  $(x, y, z)$  onto the  $xy$ -plane is the point  $(x, y, 0)$ . Similarly, the projection of this point onto the  $yz$ -plane is the point  $(0, y, z)$ , and the projection of this point onto the  $xz$ -plane is the point  $(x, 0, z)$ . Figure 1 illustrates these projections, and how they can be used to plot a point in  $xyz$ -space. One can first plot the point's projections, which is similar to the task of plotting points in the  $xy$ -plane, and then use line segments originating from these projections and perpendicular to the coordinate planes to "locate" the point in  $xyz$ -space.

## The Distance Formula

The distance between two points  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$  in the  $xy$ -plane is given by the distance formula,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Similarly, the distance between two points  $P_1 = (x_1, y_1, z_1)$  and  $P_2 = (x_2, y_2, z_2)$  in  $xyz$ -space is given by the following generalization of the distance formula,

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

This can be proved by repeated application of the Pythagorean Theorem.

**Example** The distance between  $P_1 = (2, 3, 1)$  and  $P_2 = (8, -5, 0)$  is

$$d(P_1, P_2) = \sqrt{(8 - 2)^2 + (-5 - 3)^2 + (0 - 1)^2} = \sqrt{36 + 64 + 1} = \sqrt{101} \approx 10.05.$$

□

## Equations of Surfaces

In two dimensions, the solution set of a single equation involving the coordinates  $x$  and/or  $y$  is a curve. In three dimensions, the solution set of an equation involving  $x$ ,  $y$  and/or  $z$  is a surface.

**Example** The equation  $z = 3$  describes a plane that is parallel to the  $xy$ -plane, and is 3 units "above" it; that is, it lies 3 units along the positive  $z$ -axis from the  $xy$ -plane. On the other hand, the equation  $x = y$  describes a plane consisting of all points whose  $x$ - and  $y$ -coordinates are equal. It is not parallel to any coordinate plane, but it contains the  $z$ -axis, which consists of all points whose  $x$ - and  $y$ -coordinates are both zero, and it intersects the  $xy$ -plane at the line  $y = x$ . □

The equation of a sphere with center  $C = (h, k, \ell)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2.$$

The *unit sphere* has center  $O = (0, 0, 0)$  and radius 1:

$$x^2 + y^2 + z^2 = 1.$$

We now illustrate how to work with equations of spheres.

**Example** The equation of a sphere with center  $C = (-3, -1, 1)$  and radius  $r = 10$  is

$$(x - (-3))^2 + (y - (-1))^2 + (z - 1)^2 = 10^2,$$

or

$$(x + 3)^2 + (y + 1)^2 + (z - 1)^2 = 100.$$

Expanding, we obtain

$$x^2 + y^2 + z^2 + 6x + 2y - 2z = 89,$$

which obscures the center and radius, but it is still possible to detect that the equation represents a sphere, due to the fact that the  $x^2$ ,  $y^2$  and  $z^2$  terms have equal coefficients.  $\square$

**Example** The equation

$$4x^2 + 4y^2 + 4z^2 - 8x - 16y - 16 = 0$$

describes a sphere, as can be seen by the equal coefficients in front of the  $x^2$ ,  $y^2$  and  $z^2$ . To determine the radius and center of the sphere, we complete the square in  $x$  and  $y$ :

$$\begin{aligned} 0 &= 4x^2 + 4y^2 + 4z^2 - 8x - 16y - 16 \\ &= 4(x^2 - 2x) + 4(y^2 - 4y) + 4z^2 - 16 \\ &= 4(x^2 - 2x + 1 - 1) + 4(y^2 - 4y + 4 - 4) + 4z^2 - 16 \\ &= 4[(x - 1)^2 - 1] + 4[(y - 2)^2 - 4] + 4z^2 - 16 \\ &= 4(x - 1)^2 + 4(y - 2)^2 + 4z^2 - 36. \end{aligned}$$

Rearranging, we obtain the standard form of the equation of the sphere:

$$(x - 1)^2 + (y - 2)^2 + z^2 = 9,$$

which reveals that the center is at the point  $C = (1, 2, 0)$ , and the radius is  $r = 3$ .  $\square$

## Summary

- The three-dimensional rectangular coordinate system is the one-to-one correspondence between each point  $P$  in three-dimensional space, or  $xyz$ -space, and an ordered triple  $(x, y, z)$  in  $\mathbb{R}^3$ . The numbers  $x$ ,  $y$  and  $z$  are the  $x$ -,  $y$ - and  $z$ -coordinates of  $P$ . The origin  $O$  is the point with coordinates  $(0, 0, 0)$ .
- The coordinate planes are: the  $xy$ -plane, the set of all points whose  $z$ -coordinate is zero; the  $yz$ -plane, the set of all points whose  $x$ -coordinate is zero; and the  $xz$ -plane, the set of all points whose  $y$ -coordinate is zero.
- The projection of a point  $P = (x, y, z)$  onto the  $xy$ -plane is the point  $(x, y, 0)$ . The projection of  $P$  onto the  $yz$ -plane is the point  $(0, y, z)$ . The projection of  $P$  onto the  $xz$ -plane is the point  $(x, 0, z)$ .
- The distance formula states that the distance between two points in  $xyz$ -space is the square root of the sum of the squares of the differences between corresponding coordinates. That is, given  $P_1 = (x_1, y_1, z_1)$  and  $P_2 = (x_2, y_2, z_2)$ , the distance between  $P_1$  and  $P_2$  is given by  $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ .
- The equation of a sphere with center  $C = (x_0, y_0, z_0)$  and radius  $r$  is  $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ .
- An equation in which  $x^2$ ,  $y^2$  and  $z^2$  have the same coefficients describes a sphere; the center and radius can be determined by completing the square in  $x$ ,  $y$  and  $z$ .