Introduction

This course is the fourth course in the calculus sequence, following MAT 167, MAT 168 and MAT 169. Its purpose is to prepare students for more advanced mathematics courses, particularly courses in mathematical programming (MAT 419), advanced engineering mathematics (MAT 430), real analysis (MAT 441), complex analysis (MAT 436), and numerical analysis (MAT 460 and 461). The course will focus on three main areas, which we briefly discuss here.

Partial Differentiation

In single-variable calculus, you learned how to compute the derivative of a function of one variable, \( y = f(x) \), with respect to its independent variable \( x \), denoted by \( \frac{dy}{dx} \). In this course, we consider functions of several variables. In most cases, the functions we use will depend on two or three variables, denoted by \( x, y \) and \( z \), corresponding to spatial dimensions.

When a function \( f(x, y, z) \), for example, depends on several variables, it is not possible to describe its rate of change with respect to those variables using a single quantity such as the derivative. Instead, this rate of change is a vector quantity, called the gradient, denoted by \( \nabla f \).

Each component of the gradient is the partial derivative of \( f \) with respect to one of its independent variables, \( x, y \) or \( z \). That is,

\[
\nabla f = \left[ \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right].
\]

For example, the partial derivative of \( f \) with respect to \( x \), denoted by \( \frac{\partial f}{\partial x} \), describes the instantaneous rate of change of \( f \) with respect to \( x \), when \( y \) and \( z \) are kept constant. Partial derivatives can be computed using the same differentiation techniques as in single-variable calculus, but one must be careful, when differentiating with respect to one variable, to treat all other variables as if they are constant. For example, if \( f(x, y) = x^2y + y^3 \), then

\[
\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 3y^2,
\]

because the \( y^3 \) term does not depend on \( x \), and therefore its partial derivative with respect to \( x \) is zero.

If

\[
F(x, y, z) = \begin{bmatrix}
F_1(x, y, z) \\
F_2(x, y, z) \\
F_3(x, y, z)
\end{bmatrix}
\]
is a vector-valued function of three variables, then each of its component functions \( F_1, F_2, \) and \( F_3 \) has a gradient vector, and the rate of change of \( \mathbf{F} \) with respect to \( x, y \) and \( z \) is described by a matrix, called the Jacobian matrix

\[
J_F(x, y, z) = \begin{bmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\
\frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z}
\end{bmatrix},
\]

where each entry of \( J_F(x, y, z) \) is a partial derivative of one of the component functions with respect to one of the independent variables.

We will learn how to generalize various concepts and techniques from single-variable differential calculus to the multi-variable case. These include:

- tangent lines, which become tangent planes for functions of two variables and tangent spaces for functions of three or more variables. These are used to compute linear approximations similar to those of functions of a single variable.

- The Chain Rule, which generalizes from a product of derivatives to a product of Jacobian matrices, using standard matrix multiplication. This allows computing the rate of change of a function as its independent variables change along any direction in space, not just along any of the coordinate axes, which in turn allows determination of the direction in which a function increases or decreases most rapidly.

- computing maximum and minimum values of functions, which, in the multi-variable case, requires finding points at which the gradient is equal to the zero vector (corresponding to finding points at which the derivative is equal to zero) and checking whether the matrix of second partial derivatives is positive definite for a minimum, or negative definite for a maximum (which generalizes the second derivative test from the single-variable case). We will also learn how to compute maximum and minimum values subject to constraints on the independent variables, using the method of Lagrange multipliers.

Multiple Integration

Next, we will learn how to compute integrals of functions of several variables over multi-dimensional domains, generalizing the definite integral of a function \( f(x) \) over an interval \([a, b]\). The integral of a function of two variables \( f(x, y) \) represents the volume under a surface described by the graph of \( f \), just as the integral of \( f(x) \) is the area under the curve described by the graph of \( f \).

In some cases, it is more convenient to evaluate an integral by first performing a change of variables, as in the single-variable case. For example, when integrating a function of two variables, polar coordinates is useful. For functions of three variables, cylindrical and spherical coordinates, which are both generalizations of polar coordinates, are worth considering.
In the general case, evaluating the integral of a function of \( n \) variables by first changing to \( n \) different variables requires multiplying the integrand by the determinant of the Jacobian matrix of the function that maps the new variables to the old. This is a generalization of the \( u \)-substitution from single-variable calculus, and also relates to formulas for area and volume from MAT 169 that are defined in terms of determinants, or equivalently, in terms of the dot product and cross product.

**Vector Calculus**

In the last part of the course, we will study *vector fields*, which are functions that assign a vector to each point in its domain, like the vector-valued function \( \mathbf{F} \) described above. We will first learn how to compute *line integrals*, which are integrals of functions along curves. A line integral can be viewed as a generalization of the integral of a function on an interval, in that \( dx \) is replaced by \( ds \), an infinitesimal distance between points on the curve. It can also be viewed as a generalization of an integral that computes the arc length of a curve, as the line integral of a function that is equal to one yields the arc length. A line integral of a vector field is useful for computing the work done by a force applied to an object to move it along a curved path. To facilitate the computation of line integrals, a variation of the Fundamental Theorem of Calculus is introduced.

Next, we generalize the notion of a parametric curve to a parametric surface, in which the coordinates of points on the surface depend on two parameters \( u \) and \( v \), instead of a single parameter \( t \) for a parametric curve. Using the relation between the cross product and the area of a parallelogram, we can define the integral of a function over a parametric surface, which is similar to how a change of variables in a double integral is handled. Then, we will learn how to integrate vector fields over parametric surfaces, which is useful for computing the mass of fluid that crosses a surface, given the rate of flow per unit area.

We conclude with discussion of several fundamental theorems of vector calculus: Green’s Theorem, Stokes’ Theorem, and the Divergence Theorem. All of these can be seen to be generalizations of the Fundamental Theorem of Calculus to higher dimensions, in that they relate the integral of a function over the interior of a domain to an integral of a related function over its boundary. These theorems can be conveniently stated using the div and curl operations on vector fields. Specifically, if \( \mathbf{F} = \langle P, Q, R \rangle \), then

\[
\text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},
\]

\[
\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.
\]

However, using the language of *differential forms*, we can condense the Fundamental Theorem of Calculus and all four of its variations into one theorem, known as the General Stokes’ Theorem. All six results are stated on the next page.
Fundamental Theorem of Calculus:
\[ \int_a^b f'(x) \, dx = f(b) - f(a) \]
where \( f \) is continuously differentiable on \([a, b]\)

Fundamental Theorem of Line Integrals:
\[ \int_a^b \nabla f(r(t)) \cdot r'(t) \, dt = f(r(b)) - f(r(a)) \]
where \( r(t) = \langle x(t), y(t), z(t) \rangle \), \( a \leq t \leq b \), is the position function for a curve \( C \) and \( f(x, y, z) \) is a continuously differentiable function defined on \( C \)

Green’s Theorem:
\[ \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{\partial D} P \, dx + Q \, dy \]
where \( D \) is a 2-D region with piecewise smooth boundary \( \partial D \) and \( P \) and \( Q \) are continuously differentiable on \( D \)

Stokes’ Theorem:
\[ \int_S \text{curl} \mathbf{F} \cdot \mathbf{n} ds = \int_{\partial S} \mathbf{F} \cdot \mathbf{T} ds \]
where \( S \) is a surface in 3-D with unit normal vector \( \mathbf{n} \), and piecewise smooth boundary \( \partial S \) with unit tangent vector \( \mathbf{T} \), and \( \mathbf{F} \) is a continuously differentiable vector field

Divergence Theorem:
\[ \int_E \text{div} \mathbf{F} \, dV = \int_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS \]
where \( E \) is a solid region in 3-D with boundary surface \( \partial E \), which has outward unit normal vector \( \mathbf{n} \), and \( \mathbf{F} \) is a continuously differentiable vector field

General Stokes’ Theorem:
\[ \int_M d\omega = \int_{\partial M} \omega \]
where \( M \) is a \( k \)-manifold and \( \omega \) is a \((k - 1)\)-form on \( M \)