These notes correspond to Section 13.1 in Stewart and Section 4.3 in Marsden and Tromba.

Vector Fields

To this point, we have mostly worked with scalar-valued functions of several variables, in the interest of computing quantities such as the maximum or minimum value of a function, or the volume or center of mass of a solid. Now, we will study applications involving *vector-valued* functions of several variables. The difficulty of visualizing such functions leads to the notion of a *vector field*.

A function \( F: U \subseteq \mathbb{R}^n \to \mathbb{R}^n \) is a function that assigns to each point \( x \in U \) a vector 

\[
F(x) = \langle F_1(x), F_2(x), \ldots, F_n(x) \rangle
\]

in \( \mathbb{R}^n \). The functions \( F_1, F_2, \ldots, F_n \) are the *component functions*, or *component scalar fields*, of \( F \). For our purposes, \( n = 2 \) or \( 3 \). To visualize a vector field, one can plot the vector \( F(x) \) at any given point \( x \), using the component functions to obtain the components of the vector to be plotted at each point.

The following are certain vector fields of interest in applications:

- Given a fluid, for example, a *velocity field* is a vector field \( V(x, y, z) \) that indicates the velocity of the fluid at each point \( (x, y, z) \). When plotting a velocity field, the speed of the fluid at each point is indicated by the length of the vector plotted at that point, and the direction of the fluid at that point is indicated by the direction of the vector.

A curve \( c(t) \) is said to be a *flow line*, or *streamline*, of a velocity field \( V \) if, for each value of the parameter \( t \),

\[
c'(t) = V(c(t)).
\]

That is, at each point along the curve, its tangent vector coincides with \( V \). A flow line can be approximated by first choosing an initial point \( x_0 = c(t_0) \), then using the value of \( V \) at that point to approximate a second point \( x_1 = c(t_1) \) as follows:

\[
\frac{x_1 - x_0}{t_1 - t_0} = \frac{c(t_1) - c(t_0)}{t_1 - t_0} \approx V(c(t_0)) \implies x_1 \approx x_0 + (t_1 - t_0)V(x_0).
\]

This can be continued to obtain the locations of any number of points along the flow line. The closer the times \( t_0, t_1, \ldots \) are to one another, the more accurate the approximate flow line will be.
Consider two objects with mass $m$ and $M$, with the object of mass $M$ located at the origin, and the vector field $\mathbf{F}$ defined by

$$\mathbf{F}(\mathbf{r}) = -\frac{mMG}{\|\mathbf{r}\|^3} \mathbf{r},$$

where $\mathbf{r}$ is a position vector of the object of mass $m$, and $G$ is the gravitational constant. This vector field indicates the gravitational force exerted by the object at the origin on the object at position $\mathbf{r}$, and is therefore an example of a gravitational field.

Suppose an electric charge $Q$ is located at the origin, and a charge $q$ is located at the point with position vector $\mathbf{x}$. Then the electric force exerted by the first charge on the second is given by the vector field

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon qQ}{\|\mathbf{x}\|^3} \mathbf{x},$$

where $\varepsilon$ is a constant. This field, and the gravitational field described above, are both examples of force fields.
Figure 2: The conservative vector field $\mathbf{F}(x, y) = \langle y, x \rangle$

- A vector field $\mathbf{F}$ is said to be conservative if $\mathbf{F} = \nabla f$ for some function $f$. We also say that $\mathbf{F}$ is a gradient field, and $f$ is a potential function for $\mathbf{F}$. When we discuss line integrals, we will learn the physical meaning of a conservative vector field.

In upcoming lectures we will learn how to integrate vector fields, as well as the physical interpretations of such integrals.

**Example** Consider the velocity field $\mathbf{V}(x, y) = \langle -y, x \rangle$. It is shown in Figure 1. It can be seen from the figure that the flow lines of this velocity field are circles centered at the origin. □

**Example** The vector field $\mathbf{F}(x, y) = \langle y, x \rangle$ is conservative, because $\mathbf{F} = \nabla f$, where $f(x, y) = xy$. The field is shown in Figure 2. It should be noted that conservative vector fields are also called irrotational; a fluid whose velocity field is conservative has no vorticity. □
Practice Problems

Practice problems from the recommended textbooks are:

- Stewart: Section 13.1, Exercises 1-17 odd, 21-25 odd, 29, 31
- Marsden/Tromba: Section 4.3, Exercises 1-15 odd