These notes correspond to Section 11.1 in Stewart and Section 2.1 in Marsden and Tromba.

Functions of Several Variables

Multi-variable calculus is concerned with the study of quantities that depend on more than one variable, such as temperature that varies within a three-dimensional object, which is a scalar quantity, or the velocity of a flowing liquid, which is a vector quantity. To aid in this study, we first introduce some important terminology and notation that is useful for working with functions of more than one variable, and then introduce some techniques for visualizing such functions.

Terminology and Notation for Functions and their Graphs

The following standard notation and terminology is used to define, and discuss, functions of several variables and their visual representations. As they will be used throughout the course, it is important to become acquainted with them immediately.

• The set $\mathbb{R}$ is the set of all real numbers.

• If $S$ is a set and $x$ is an element of $S$, we write $x \in S$.

  \textbf{Example} $2 \in \mathbb{R}$, and $\pi \in \mathbb{R}$, but $i \notin \mathbb{R}$, where $i = \sqrt{-1}$ is an imaginary number.

• If $S$ is a set and $T$ is a subset of $S$ (that is, every element of $T$ is in $S$), we write $T \subseteq S$.

  \textbf{Example} The set of natural numbers, $\mathbb{N}$, and the set of integers, $\mathbb{Z}$, satisfy $\mathbb{N} \subseteq \mathbb{Z}$. Furthermore, $\mathbb{Z} \subseteq \mathbb{R}$.

• The set $\mathbb{R}^n$ is the set of all ordered $n$-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers. Each real number $x_i$ is called a coordinate of the point $x$.

  \textbf{Example} Each point $(x, y) \in \mathbb{R}^2$ has coordinates $x$ and $y$. Each point $(x, y, z) \in \mathbb{R}^3$ has an $x$-, $y$- and $z$-coordinate.

• A \textit{function} $f$ with \textit{domain} $D \subseteq \mathbb{R}^n$ and \textit{range} $R \subseteq \mathbb{R}^m$ is a set of ordered pairs of the form $(x, y)$, where $x \in D$ and $y \in R$, such that each element $x \in D$ is mapped to only one element of $R$. That is, there is only one ordered pair in $f$ such that $x$ is its first element. We write $f : D \rightarrow R$ to indicate that $f$ maps elements of $D$ to elements of $R$. We also say that $f$ maps $D$ into $R$. 
Example Let $R^+$ denote the set of non-negative real numbers. The function $f(x, y) = x^2 + y^2$ maps $\mathbb{R}^2$ into $R^+$, and we can write $f : \mathbb{R}^2 \to R^+$.

- Let $f : D \to R$, and let $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}^m$. If $m = 1$, we say that $f$ is a scalar-valued function, and if $m > 1$, we say that $f$ is a vector-valued function. If $n = 1$, we say that $f$ is a function of one variable, and if $n > 1$, we say that $f$ is a function of several variables. For each $x = (x_1, x_2, \ldots, x_n) \in D$, the coordinates $x_1, x_2, \ldots, x_n$ of $x$ are called the independent variables of $f$, and for each $y = (y_1, y_2, \ldots, y_m) \in R$, the coordinates $y_1, y_2, \ldots, y_m$ of $y$ are called the dependent variables of $f$.

Example The function $z = x^2 + y^2$ is a scalar-valued function of several variables. The independent variables are $x$ and $y$, and the dependent variable is $z$. The function $r(t) = (x(t), y(t), z(t))$, where $x(t) = t \cos t$, $y(t) = t \sin t$, and $z(t) = e^t$, is a vector-valued function with independent variable $t$ and dependent variables $x$, $y$, and $z$.

- Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. The graph of $f$ is the subset of $\mathbb{R}^{n+1}$ consisting of the points $(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$, where $(x_1, x_2, \ldots, x_n) \in D$.

Solution The graph of the function $z = x^2 + y^2$ is a parabola of revolution obtained by revolving the parabola $z = x^2$ around the $z$-axis. The graph of the function $z = x + y - 1$ is a line in 3-D space that passes through the points $(0, 0, -1)$ and $(1, 1, 1)$.

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function if $f$ has the form
  \[ f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b, \]
  where $a_1, a_2, \ldots, a_n$ and $b$ are constants.

Example The function $y = mx + b$ is a linear function of the single independent variable $x$. Its graph is a line contained within the $xy$-plane, with slope $m$, passing through the point $(0, b)$. The function $z = ax + by + c$ is a linear function of the two independent variables $x$ and $y$. Its graph is a line in 3-D space that passes through the points $(0, 0, c)$ and $(1, 1, a + b + c)$.

- Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that a set $L$ is a level set of $f$ if $L \subseteq D$ and $f$ is equal to a constant value $k$ on $L$; that is, $f(x) = k$ if $x \in L$. If $n = 2$, we say that $L$ is a level curve or level contour; if $n = 3$, we say that $L$ is a level surface.

Example A level surface of the function $f(x, y, z) = x^2 + y^2 + z^2$, where $f(x, y, z) = k$ for a constant $k$, is a sphere of radius $\sqrt{k}$. The level curves of the function $z = x^2 + y^2$ are circles of radius $\sqrt{k}$ with center $(0, 0, k)$, situated in the plane $z = k$, for each nonnegative number $k$.

Visualization Techniques

While it is always possible to obtain the graph of a function $f(x, y)$, for example, by substituting various values for its independent variables and plotting the corresponding points from the graph,
this approach is not necessarily helpful for understanding the graph as a whole. Knowing the extent of the possible values of a function’s independent and dependent variables (the domain and range, respectively), along with the behavior of a few select curves that are contained within the function’s graph, can be more helpful. To that end, we mention the following useful techniques for acquiring this information.

To find the domain and range of a function $f$, it is often necessary to account for the domains of functions that are included in the definition of $f$. For example, if there is a square root, it is necessary to avoid taking the square root of a negative number.

**Example** Let $f(x, y) = \ln|x^2 - y^2|$. Since $\ln|x|$ is only defined for $x > 0$, we must have $x^2 > y^2$, which, upon taking the square root of both sides, yields $|x| > |y|$. Therefore, this inequality defines the domain of $f$. The range of $f$ is the range of $\ln$, which is $\mathbb{R}$.

To find the level set of a function $f(x_1, x_2, \ldots, x_n)$, solve the equation $f(x_1, x_2, \ldots, x_n) = k$, where $k$ is a constant. This equation will implicitly define the level set. In some cases, it can be solved for one of the independent variables to obtain an explicit function that describes the level set.

**Example** Let $z = 2x + y$ be a function of two variables. The graph of this function is a plane. Each level set of this function is described by an equation of the form $2x + y = k$, where $k$ is a constant. Since $z = k$ as well, the graph of this level set is the line with equation $y = -2x + k$, contained within the plane $z = k$.

**Example** Let $z = \ln y - x$. Each level set of this function is described by an equation of the form $\ln y - x = k$, where $k$ is a constant. Exponentiating both sides, we obtain $y = e^{x+k}$. It
follows that the graph of this level set is that of the exponential function, contained within the plane $z = k$, and shifted $k$ units to the left (that is, along the $x$-axis).

To help visualize a function of two variables $z = f(x, y)$, it can be helpful to use the method of sections. This involves viewing the functions when restricted to “vertical” planes, such as the $xz$-plane and the $yz$-plane. To take these two sections, first set $y = 0$ to obtain $z$ as a function of $x$, and then graph that function in the $xz$-plane. Then, set $x = 0$ to obtain $z$ as a function of $y$, and graph that function in the $yz$-plane. Using these graphs as guides, in conjunction with level curves, it is then easier to visualize what the rest of the graph of $f$ looks like.

**Example** Let $z = x^2 + y^4$. Setting $y = 0$ yields $z = x^2$, the graph of which is a parabola in the $xz$-plane. Setting $x = 0$ yields $z = y^4$, which has a graph that is a parabola-like curve, where $z$ increases much more rapidly. Combining these graphs with selected level curves, which are described by the equations $y = \sqrt{k - x^2}$, where $|x| \leq \sqrt{k}$ for $k \geq 0$, allows us to visualize the graph of this function. Level curves and sections are shown in Figures 1 and 2, respectively.
Practice Problems

1. Find the domain and range of the function \( f(x, y, z) = \sqrt{xyz^2} \).

2. Find the domain and range of the function \( f(x, y) = \ln|x^2 + y^2 - 4| \).

3. Let \( f(x, y) = y + x^2 \). Sketch the level curves \( f(x, y) = k \), for \( k = 0, 1, 2, 3 \), in the \( xy \)-plane. Use these level curves to sketch the graph of \( f \) in 3-D space.

4. Describe the level curves of the function \( f(x, y) = \ln|x^2 + 4y^2| \). Use this description to sketch the graph of \( f \) in 3-D space.

5. Sketch two sections of the function \( f(x, y) = |x| + |y| \) in 3-D space, and at least two level curves in the \( xy \)-plane. Use this information to sketch the graph of \( f \) in 3-D space.

6. Repeat the previous problem with \( f(x, y) = |x + y| \).