Limits and Continuity

Recall that in single-variable calculus, the fundamental concept of a limit was used to define derivatives and integrals of functions, as well as the notion of continuity of a function. We now generalize limits and continuity to the case of functions of several variables.

Terminology and Notation for Limits and Continuity

- Let \( f : D \subseteq \mathbb{R} \rightarrow \mathbb{R} \), and \( a \in D \). We say \( f(x) \) approaches \( L \) as \( x \) approaches \( a \), and write
  \[
  \lim_{x \to a} f(x) = L
  \]
  if, for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that if \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \epsilon \).

- If \( x = (x_1, x_2, \ldots, x_n) \) is a point in \( \mathbb{R}^n \), or, equivalently, if \( x = \langle x_1, x_2, \ldots, x_n \rangle \) is a position vector in \( \mathbb{R}^n \), then the magnitude, or length, of \( x \), denoted by \( \|x\| \), is defined by
  \[
  \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}.
  \]
  Note that if \( n = 1 \), then \( x \) is actually a scalar \( x \), and \( \|x\| = |x| \).

Example If \( x = (3, -1, 4) \in \mathbb{R}^3 \), then \( \|x\| = \sqrt{3^2 + (-1)^2 + 4^2} = \sqrt{26} \).

- Let \( f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \), and \( a \in D \). We say \( f(x) \) approaches \( b \) as \( x \) approaches \( a \), and write
  \[
  \lim_{x \to a} f(x) = b,
  \]
  if, for any \( \epsilon > 0 \), no matter how small, there exists a \( \delta > 0 \) such that for any \( x \) such that
  \[0 < \|x - a\| < \delta, \|f(x) - b\| < \epsilon.\]
  This definition is illustrated in Figure 1. Note that the condition \( \|x - a\| > 0 \) specifically excludes consideration of \( x = a \), because limits are used to understand the behavior of a function near a point, not at a point.

Example Consider the function
  \[
  f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}.
  \]
Figure 1: Illustration of the limit, as \( x \) approaches \( a \) (left plot), of \( f(x) \) being equal to \( b \) (right plot). For any ball around the point \( b \) of radius \( \epsilon \) (right plot), no matter how small, there exists a ball around the point \( a \), of radius \( \delta \) (left plot), such that every point in the ball around \( a \) is mapped by \( f \) to a point in the ball around \( b \).

We will use the definition of a limit to show that as \((x, y) \to (0, 0)\), \( f(x, y) \to 0 \). Let \( \epsilon > 0 \). We need to show that there exists some \( \delta > 0 \) such that if \( 0 < \| (x, y) - (0, 0) \| = \sqrt{x^2 + y^2} < \delta \), then \(|f(x, y) - 0| = \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| < \epsilon \). First, we note that

\[
\left| \frac{y}{\sqrt{x^2 + y^2}} \right| < \frac{y}{\sqrt{y^2}} = \frac{|y|}{|y|} = 1.
\]

Therefore, if we set \( \delta = \epsilon \), we obtain

\[
\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = |x| \left| \frac{y}{\sqrt{x^2 + y^2}} \right| < |x| = \sqrt{x^2} < \sqrt{x^2 + y^2} < \delta = \epsilon,
\]

from which it follows that the limit exists and is equal to zero. \( \Box \)

- Let \( f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m \), and let \( a \in D \). We say that \( f \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \).
- Let \( f : D \subseteq \mathbb{R}^n \to \mathbb{R} \). We say that \( f \) is a polynomial if, for each \( x = (x_1, x_2, \ldots, x_n) \in D \), \( f(x) \) is equal to a sum of terms of the form \( x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \), where \( p_1, p_2, \ldots, p_n \) are nonnegative integers.
Example The functions \( f(x) = x^3 + 3x^2 + 2x + 1 \), \( g(x, y) = x^2y^3 + 3xy + x^2 + 1 \), and \( h(x, y, z) = 4xy^2z^3 + 8yz^2 \) are all examples of polynomials. □

- Let \( f, p, q : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \), and let \( q(x) \neq 0 \) on \( D \). We say that \( f \) is a rational function if \( p \) and \( q \) are both polynomials and \( f(x) = p(x)/q(x) \).

Example The functions \( f(x) = 1/(x + 1) \), \( g(x, y) = xy^2/(x^2 + y^3) \), and \( h(x, y, z) = (xy^2 + z^3)/(x^2z + xyz^2 + yz^3) \) are all examples of rational functions. □

- An algebraic function is a function that satisfies a polynomial equation whose coefficients are themselves polynomials.

Example The square root function \( y = \sqrt{x} \) is an algebraic function, because it satisfies the equation \( y^2 - x = 0 \). The function \( y = x^{5/2} + x^{3/2} \) is also an algebraic function, because it satisfies the equation \( x^5 + 2x^4 + x^3 - y^2 = 0 \). □

**Defining Limits Using Neighborhoods**

An alternative approach to defining limits involves the concept of a neighborhood, which generalizes open intervals on the real number line.

- Let \( x_0 \in \mathbb{R}^n \) and let \( r > 0 \). We define the ball centered at \( x_0 \) of radius \( r \), denoted by \( D_r(x_0) \), to be the set of all points \( x \in \mathbb{R}^n \) such that \( ||x - x_0|| < r \).

Example In 1-D, the open interval \((0, 1)\) is also the ball centered at \( x_0 = 1/2 \) of radius \( r = 1/2 \). In 3-D, the inside of the sphere with center \((0, 0, 0)\) and radius 2, \( \{(x, y, z) | x^2 + y^2 + z^2 < 4\} \), is also the ball \( D_2((0, 0, 0)) \). □

- We say that a set \( U \subseteq \mathbb{R}^n \) is open if, for any point \( x_0 \in U \), there exists an \( r > 0 \) such that \( D_r(x_0) \subseteq U \).

Example In 1-D, any open set is an open interval, such as \((-1, 1)\), or a union of open intervals. In 2-D, the interior of the ellipse defined by the equation \( 4x^2 + 9y^2 = 1 \) is an open set; the ellipse itself is not included. □

- Let \( x_0 \in \mathbb{R}^n \). We say that \( N \) is a neighborhood of \( x_0 \) if \( N \) is an open set that contains \( x_0 \).

- Let \( A \subseteq \mathbb{R}^n \) be an open set. We say that \( x_0 \in \mathbb{R}^n \) is a boundary point of \( A \) if every neighborhood of \( x_0 \) contains at least one point in \( A \) and one point not in \( A \).

Example Let \( D = \{(x, y) | x^2 + y^2 < 1\} \), which is often called the unit ball in \( \mathbb{R}^2 \). This set consists of all points inside the unit circle with center \((0, 0)\) and radius 1, not including the circle itself. The point \((x_0, y_0) = (\sqrt{2}/2, \sqrt{2}/2)\), which is on the circle, is a boundary point of \( D \) because, as illustrated in Figure 2, any neighborhood of \((x_0, y_0)\) must contain points inside the circle, and points that are outside. □
Figure 2: Boundary point \((x_0, y_0)\) of the set \(D = \{(x, y) | x^2 + y^2 < 1\}\). The neighborhood of \((x_0, y_0)\) shown, \(D_r((x_0, y_0)) = \{(x, y) | (x - x_0)^2 + (y - y_0)^2 < 0.1\}\), contains points that are in \(D\) and points that are not in \(D\).

- Let \(f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m\), and let \(a \in D\) or let \(a\) be a boundary point of \(D\). We say that \(a \in D\). We say that \(f(x)\) approaches \(b\) as \(x\) approaches \(a\), and write
  \[
  \lim_{x \to a} f(x) = b,
  \]
  if, for any neighborhood \(N\) of \(b\), there exists a neighborhood \(U\) of \(a\) such that if \(x \in U\), then \(f(x) \in N\).

Results

In the statement of the following results, \(f, g : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m\), \(a \in D\) or \(a\) is a boundary point of \(D\), \(b, b_1, b_2 \in \mathbb{R}^m\), and \(c \in \mathbb{R}\).
The limit of a function \( f(x) \) as \( x \) approaches \( a \), if it exists, is unique. That is, if
\[
\lim_{x \to a} f(x) = b_1 \quad \text{and} \quad \lim_{x \to a} f(x) = b_2,
\]
then \( b_1 = b_2 \). It follows that if \( f(x) \) approaches two distinct values as \( x \) approaches \( a \) along two distinct paths, then the limit as \( x \) approaches \( a \) does not exist.

- If \( \lim_{x \to a} f(x) = b \), then \( \lim_{x \to a} cf(x) = cb \).
- If \( \lim_{x \to a} f(x) = b_1 \) and \( \lim_{x \to a} g(x) = b_2 \), then
  \[
  \lim_{x \to a} (f + g)(x) = b_1 + b_2.
  \]
  Furthermore, if \( m = 1 \), then
  \[
  \lim_{x \to a} (fg)(x) = b_1b_2.
  \]
- If \( m = 1 \) and \( \lim_{x \to a} f(x) = b \neq 0 \), and \( f(x) \neq 0 \) in a neighborhood of \( a \), then
  \[
  \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{b}.
  \]
- If \( f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \), where \( f_1, f_2, \ldots, f_m \) are the component functions of \( f \), and \( b = (b_1, b_2, \ldots, b_m) \), then \( \lim_{x \to a} f(x) = b \) if and only if \( \lim_{x \to a} f_i(x) = b_i \) for \( i = 1, 2, \ldots, m \).
- If \( f \) and \( g \) are continuous at \( a \), then so is \( cf \) and \( f + g \). If, in addition, \( m = 1 \), then \( fg \) is continuous at \( a \). Furthermore, if \( m = 1 \) and if \( f \) is nonzero in a neighborhood of \( a \), then \( 1/f \) is continuous at \( a \).
- If \( f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \), where \( f_1, f_2, \ldots, f_m \) are the component functions of \( f \), then \( f \) is continuous at \( a \) if and only if \( f_i \) is continuous at \( a \) for \( i = 1, 2, \ldots, m \).
- Any polynomial function \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous on all of \( \mathbb{R}^n \).
- Any rational function \( f : D \subseteq \mathbb{R}^n \to \mathbb{R} \) is continuous wherever it is defined.

**Example** The function \( f(x, y) = 2x/(x^2 - y^2) \) is defined on all of \( \mathbb{R}^2 \) except where \( x^2 - y^2 = 0 \); that is, where \( |x| = |y| \). Therefore, \( f \) is continuous at all such points. □

- Let \( f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m \), and let \( g : U \subseteq \mathbb{R}^p \to D \). If the composition \( (f \circ g)(x) = f(g(x)) \) defined on \( U \), then \( f \circ g \) is continuous at \( a \in U \) if \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \).

**Example** The function \( g(x, y) = x^2 + y^2 \), being a polynomial, is continuous on all of \( \mathbb{R}^2 \). The function \( f(z) = \sin z \) is continuous on all of \( \mathbb{R} \). Therefore, the composition \( (f \circ g)(x, y) = f(g(x, y)) = \sin(x^2 + y^2) \) is continuous on all of \( \mathbb{R}^2 \). □

- Algebraic functions, such as \( x^r \) where \( r \) is any rational number (for example, \( f(x) = \sqrt{x} \)) and trigonometric functions, such as \( \sin x \) or \( \tan x \), are continuous wherever they are defined.
Techniques for Establishing Limits and Continuity

We now discuss some techniques for computing limits of functions of several variables, or determining that they do not exist. We also demonstrate how to determine if a function is continuous at a point.

• To show that the limit of a function \( f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) as \( x \rightarrow a \) does not exist, try letting \( x \) approach \( a \) along different paths to see if different values are approached. If they are, then the limit does not exist.

For example, let \( n = 2 \) and let \( x = (x, y) \) and \( a = (a_1, a_2) \). Then, try setting \( x = a_1 \) in the formula for \( f(x, y) \) and letting \( y \) approach \( a_2 \), or vice versa. Other possible paths include, for example, setting \( x = cy \), where \( c = a_1/a_2 \), if \( a_2 \neq 0 \), and letting \( y \) approach \( a_2 \), or considering the cases of \( x < a_1 \) and \( x > a_1 \), or \( y < a_2 \) and \( y > a_2 \), separately.

**Example** Let \( f(x, y) = x^3y/(x^4 + y^4) \). If we let \( (x, y) \rightarrow (0, 0) \) by first setting \( y = 0 \) and then letting \( x \rightarrow 0 \), we observe that \( f(x, 0) = x^3(0)/(x^4 + 0) = 0 \) for all \( x \neq 0 \). This suggests that \( f(x, y) \rightarrow 0 \) as \( (x, y) \rightarrow (0, 0) \). However, if we set \( x = y \) and let \( x, y \rightarrow 0 \) together, we note that \( f(x, x) = x^3x/(x^4 + x^4) = x^4/(2x^4) = 1/2 \), which suggests that the limit is equal to \( 1/2 \). We conclude that the limit does not exist. \( \square \)

• To show that the limit of a function \( f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) as \( x \rightarrow a \) does exist and is equal to \( b \), use the definition of a limit by first assuming \( \epsilon > 0 \), and then trying to find a \( \delta \) so that \( |f(x) - b| < \epsilon \) whenever \( 0 < ||x - a|| < \delta \).

To that end, try to find an *upper bound* on \( |f(x) - b| \) in terms of \( ||x - a|| \). Specifically, if it can be shown that \( |f(x) - b| < g(||x - a||) \), where \( g \) is an invertible, increasing function, then a suitable choice for \( \delta \) is \( \delta = g^{-1}(\epsilon) \). Then, if \( ||x - a|| < \delta = g^{-1}(\epsilon) \), then

\[
|f(x) - b| < g(||x - a||) < g(g^{-1}(\epsilon)) = \epsilon.
\]

**Example** Let \( f(x, y) = (x^3 - y^3)/(x^2 + y^2) \). Letting \( (x, y) \rightarrow (0, 0) \) along various paths, it appears that \( f(x, y) \rightarrow 0 \) as \( (x, y) \rightarrow (0, 0) \). To confirm this, we assume \( \epsilon > 0 \) and try to find \( \delta > 0 \) such that if \( 0 < \sqrt{x^2 + y^2} < \delta \), then \( |(x^3 - y^3)/(x^2 + y^2)| < \epsilon \).

Factoring the numerator of \( f(x, y) \), we obtain

\[
\left| \frac{x^3 - y^3}{x^2 + y^2} \right| = \left| \frac{(x - y)(x^2 + xy + y^2)}{x^2 + y^2} \right| = \left| (x - y) \left(1 + \frac{xy}{x^2 + y^2}\right) \right|.
\]

Using \( |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} \), and similarly \( |y| \leq \sqrt{x^2 + y^2} \), yields

\[
\left| \frac{x^3 - y^3}{x^2 + y^2} \right| = |x - y| \left| 1 + \left( \frac{x}{\sqrt{x^2 + y^2}} \right) \left( \frac{y}{\sqrt{x^2 + y^2}} \right) \right| \leq 2|x - y| \leq 2(|x| + |y|) \leq 4\sqrt{x^2 + y^2}.
\]
Therefore, if we let $\delta = \epsilon / 4$, it follows that when $\sqrt{x^2 + y^2} < \delta$, then $|f(x, y)| < 4\delta = 4(\epsilon / 4) = \epsilon$, and therefore the limit exists and is equal to zero. □

- Sometimes, it is helpful to simplify the formula for a function before attempting to determine its limit.

**Example** Consider the function

$$f(x, y) = \frac{(x + y)^2 - (x - y)^2}{xy}, \quad x, y \neq 0.$$ 

Expanding the numerator yields, for $x, y \neq 0$,

$$f(x, y) = \frac{(x^2 + 2xy + y^2) - (x^2 - 2xy + y^2)}{xy} = \frac{4xy}{xy} = 4.$$ 

Therefore, even though $f(x, y)$ is not defined at $(0, 0)$, its limit as $(x, y) \to (0, 0)$ exists, and is equal to 4. This example demonstrates that a limit depends only on the behavior of a function near a particular point; what happens at that point is irrelevant. □

- In many cases, determining whether a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous can be accomplished by applying the various properties of continuous functions stated above, and using the fact that various types of functions, such as polynomial and rational functions, are known to be continuous wherever they are defined.

**Example** Let $c = \langle 2, -1, 3 \rangle$, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x) = c \times x$, the cross product of the vector $c$ and the vector $x = \langle x_1, x_2, x_3 \rangle$. This function is continuous on all of $\mathbb{R}^3$, because

$$f(x) = c \times x = \langle -x_3 - 3x_2, 3x_1 - 2x_3, 2x_2 + x_1 \rangle,$$

and each component function of $f$ can be seen to be not only a polynomial, but a linear function. □

However, in cases where a function is defined in a piecewise manner, continuity at boundaries between pieces must be determined by applying the definition of continuity directly, which requires computing limits.

**Example** Consider the function

$$f(x, y) = \begin{cases} \frac{xy^2}{2x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$ 

For $(x, y) \neq (0, 0)$, $f$ is continuous at $(x, y)$ because it is a rational function that is defined. As $(x, y) \to (0, 0)$, $f(x, y) \to 0$, as can be shown by applying the definition of a limit with $\delta = \epsilon$. Because this limit is equal to $f(0, 0) = 0$, we conclude that $f$ is continuous at $(0, 0)$ as well. □
Practice Problems

1. Compute

\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^4 + y^3}, \]

if it exists.

2. Compute

\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^3 + y^3}, \]

if it exists.

3. Compute

\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2}, \]

if it exists.

4. Suppose that \( f(x,y) \) is a rational function, meaning that it has the form

\[ f(x,y) = \frac{p(x,y)}{q(x,y)}, \]

where \( p(x,y) \) and \( q(x,y) \) are both polynomials. Furthermore, suppose that \( p(0,0) = q(0,0) = 0 \). Explain how you can use the exponents of the terms in \( p(x,y) \) and \( q(x,y) \) to make an educated guess as to whether \( f(x,y) \) has a limit as \( (x,y) \to (0,0) \), or, if it does not, why it does not.

5. Show that the function \( f(x,y,z) = \sqrt{x^2 + y^2 + z^2} \) is continuous on all of \( \mathbb{R}^3 \).

6. Recall that the triple product of the vectors \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^3 \) is defined by the following function \( f : \mathbb{R}^9 \to \mathbb{R} \):

\[ f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}), \]

where \( \times \) denotes the cross product, and \( \cdot \) denotes the dot product. Show that this function is continuous on all of \( \mathbb{R}^9 \).

7. Show that the vector-valued function \( \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3 \), defined by

\[ \mathbf{f}(u,v) = \left( \frac{1}{2} + \frac{v}{2} \cos \frac{u}{2} \right) \cos u, \left( \frac{1}{2} + \frac{v}{2} \cos \frac{u}{2} \right) \sin u, \frac{v}{2} \sin \frac{u}{2} \),

is continuous on all of \( \mathbb{R}^2 \).

8. Determine where the function \( f(x,y) = \tan(\sqrt{xy}) \) is continuous.
9. Determine where the function defined by

\[ f(x, y) = \begin{cases} \frac{(1+x)^2-(1-y)^2}{x+y} & x \neq -y \\ 2 + 2x & x = -y \end{cases} \]

is continuous.

10. Suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous on all of \( \mathbb{R}^2 \). Show that \( |f| \) is also continuous on all of \( \mathbb{R}^2 \). \textit{Hint:} consider the cases of \( f(x, y) > 0 \), \( f(x, y) < 0 \), and \( f(x, y) = 0 \) separately.

\textbf{Additional Practice Problems}

Additional practice problems from the recommended textbooks are:

- Stewart: Section 11.2, Exercises 1-15 odd, 19-27 odd, 31
- Marsden/Tromba: Section 2.2, Exercises 5-15 odd